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ORIGINAL RESEARCH

Structure connectivity of folded crossed cubes based on faulty stars

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Abstract

Large parallel computer systems bounding experience faults are inevitable due to their scale sizes, which poses serious reliability challenges for interconnection networks. Two new indicators were recently introduced to assess the stability of these networks more accurately, including structure connectivity and substructure connectivity. These parameters are crucial in measuring fault tolerance during chip failures. Let H be a certain graph pattern, and $\mathcal{F}$ be a set of subgraphs in a graph G. Then, $\mathcal{F}$ is called an H-structure cut (resp. H-substructure cut) of G if every element of $\mathcal{F}$ is isomorphic to H (resp. isomorphic to a connected subgraph of H) when $G-\mathcal{F}$ is disconnected. The H-structure connectivity $\kappa(G; H)$ (resp. H-substructure connectivity $\kappa^s(G; H)$) is the minimum cardinality over all H-structure cuts (resp. H-substructure cuts). Recently, Ba, in her Ph.D. dissertation, posted the result of $K_{1,r}$-(sub)structure connectivity of FCQ_n for $1 \le r \le \frac{n}{2}$, where FCQ_n denotes the n-dimensional folded crossed cube, which is a variant of the hypercube called crossed cube by enhancing a folded link between any two complementary vertices. In this paper, to supplement the completeness of the findings of this study, we successfully determine the $K_{1,r}$-(sub)structure connectivity of FCQ_n for $\frac{n}{2}+1 \le r \le n$, which solves the open problem proposed by Ba.

Keywords Folded crossed cubes · Structure connectivity · Interconnection network

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1 Introduction

In the study of large multiprocessor systems, various types of networks have been proposed as topologies. A network can be represented as a simple graph $G = (V(G), E(G))$, where $V(G)$ and $E(G)$ denote the sets of processors and links, respectively. One of the classic and fundamental networks is the hypercube. The *bijective connection* (BC) network, abbreviated as BC_n, is a type of cube-based network that includes several well-known interconnection networks, such as hypercube Q_n, crossed cube CQ_n [14], twisted hypercube TQ_n [15], Möbius cube MQ_n [12], locally twisted cube LTQ_n [57], generalized cube $G(m_\ell, m_{\ell-1}, \ldots, m_1)$ [2], spined cubes SQ_n [66] and so on. In addition, some cube-based networks (rather than BC networks) including augmented cube AQ_n [11], augmented k-ary n-cube $AQ_{n,k}$ [48], balanced hypercube BQ_n [44], folded cube FQ_n [16] and so on also have received wide attention. In 2002, Zhang [63] proposed a *folded crossed cube* FCQ_n based on the structure of CQ_n. This network enhances a folded link between any two complementary vertices, making it more suitable for practical applications due to its superior properties such as high performance, low-cost architecture, short diameter, and short mean internode distance.

The fault tolerance of the processor has attracted the attention of many scholars, as the failures will affect the operation of the whole system. The *connectivity* $\kappa(G)$ of a graph G, a classical parameter to measure network reliability and fault tolerance, is the minimum number of vertices whose deletion disconnects G or leaves a single vertex. However, in reality, traditional connectivity has its deficiencies because it is almost impossible for all adjacent vertices of a vertex to be faulty simultaneously. To compensate for this disadvantage, Harary [23] proposed the so-called *conditional connectivity* for which every remaining connected component possesses a specific graph-theoretic property ρ after deleting those faulty vertices. Later on, Fábrega and Fiol [17] proposed *extra connectivity*, which is one of conditional connectivity. For a vertex subset $F \subset V(G)$, if the result of $G - F$ (i.e., the removal of F from G) is disconnected such that every remaining component has at least $g + 1$ vertices, then F is called a *g-extra vertex cut*. The *g-extra connectivity* $\kappa_g(G)$ of G is the minimum cardinality over all g-extra vertex cuts.

The g-extra connectivity mainly measures the impact of a single faulty vertex separately. That is, it is generally assumed that the failure of each single vertex u is an independent event, regardless of the relatedness of its neighbors. In fact, adjacent vertices may influence each other, and the closer they are to the faulty vertex, the more likely cascading failures will occur. Particularly, with the development of science and technology, networks and subnetworks are encapsulated into chips. If one or more vertices in a chip become faulty, the whole chip may be regarded as defective. To optimize the parameters of network fault tolerance anew, Lin et al. [31] proposed structure connectivity and substructure connectivity to measure the connectedness for a specific pattern in a graph. Fixed a subgraph H of G as a pattern, a set $\mathcal{F}$ of subgraphs of G is an *H-structure cut* (resp. *H-substructure cut*) if every element of $\mathcal{F}$ is isomorphic to H (resp. isomorphic to a subgraph of H) when $G - \mathcal{F}$ is disconnected. The *H-structure connectivity* $\kappa(G; H)$ and the *H-substructure connectivity* $\kappa^s(G; H)$ are the cardinalities of a minimum H-structure cut and a minimum H-substructure cut of G, respectively. Since every graph pattern H is a substructure of itself, we have $\kappa^s(G; H) \leq \kappa(G; H)$.

Let K_n be a complete graph on n vertices, and $K_{r,s}$ be a complete bipartite graph with two parts having r vertices and s vertices, respectively. In particular, $K_{1,r}$ is the star on $r+1$ vertices (i.e., a graph with a center vertex joining r leaves), and P_ℓ (resp. C_ℓ) be the path (resp. the cycle) with ℓ vertices. In recent years, there are rich results on structure connectivity $\kappa(G; H)$ and substructure connectivity $\kappa^s(G; H)$ of networks with small order of H, such as H is isomorphic to $K_{1,r}$ (denoted as $H \cong K_{1,r}$), $H \cong P_\ell$, and $H \cong C_\ell$. As inspired by the pioneers's work on hypercubes [31], many studies focused on cube-based networks, and we summarize the currently known results in Table 1. Note that the class of *hypercube-like networks* denoted by HL_n (or refer as *bijective connection networks* denoted by BC_n) contains many variants of hypercubes, and the $K_{1,r}$-(sub)structure connectivity of HL_n has been completely determined in [30]. Therefore, subsequent research of $K_{1,r}$-(sub)structure connectivity must focus on folded, augmented, or hierarchical cube-based networks. In addition, we can refer to the results of structure connectivity and substructure connectivity for non-cube-based networks, such as arrangement graph $A_{n,k}$ [25, 60], alternating group graph AG_n [29], star graph S_n [27, 54], (n, k)-star graph $S_{n,k}$ [28], split-star network S_n^2 [64], hierarchical star network HS_n [49], bubble-sort graph B_n [50, 58], bubble-sort star graph BS_n [59], modified bubble-sort graph MB_n [46], (n, k)-bubble-sort graph $B_{n,k}$ [20], pancake graph P_n and burnt pancake graph BP_n [13], wheel network W_n [19], WK-recursive network $K(d, t)$ [56], 2-D Tours $T(m, n)$ [10], data center network DCell and BCDC [6], and Cayley graph generated by transposition trees [41].

As mentioned before, FCQ_n exhibits higher reliability, lower cost, and improved fault tolerance in performance evaluation as well. More properties of FCQ_n, including diameter, bisection width, mean internode distance, and message traffic density, can refer to [1, 63]. Hence, FCQ_n is considered a more efficient alternative when choosing a multiprocessor system. Recent research works on FCQ_n mainly include the following. Pai et al. [39] investigated the vertex-transitivity of FCQ_n. Several studies were focused on exploring diverse connectivities and diagnosabilities of FCQ_n, such as super connectivity and super edge-connectivity [9, 22], extra connectivity and extra conditional diagnosability under PMC model [21], extra edge-connectivity [42], component connectivity (resp. g-good neighbor connectivity) and component diagnosability (resp. g-good neighbor diagnosability) under PMC model and MM* model [37]. Moreover, research on the applications of algorithmic aspects for FCQ_n can be found in the literature, e.g., constructing $n+1$ edge-independent spanning trees for fault-tolerant communication [61, 62] and applying edge-disjoint Hamiltonian cycles to all-to-all broadcasting [38].

Recently, Ba [3] studied $\kappa(FCQ_n; K_{1,r})$ for $2 \leq r \leq \frac{n}{2}$. To offer more precise measurements of the fault tolerance and reliability of FCQ_n, a recommendation suggested to examine the $K_{1,r}$-(sub)structure connectivity with larger r on this network. Accordingly, we determine in this paper the exact value $\kappa(FCQ_n; K_{1,r})$ for $3 \leq r \leq n$, which complements the range of r. The main theorem is as follows.

Theorem 1 *For $n \geq 9$ and $3 \leq r \leq n$, $\kappa(FCQ_n; K_{1,r}) = \kappa^s(FCQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$.*

Table 1 The currently known results of (sub)structure-connectivity for cube-based networks

Network G	Structure H	Range of Parameters	References
	$K_{1,r}, C_4$	$1 \leq r \leq 3$	[31]
Hypercube Q_n	$K_{1,4}, P_\ell, C_{2\ell}$	$3 \leq \ell \leq n$	[43]
	$K_{1,r}$	$r \geq 2, n > f(r)^\dagger$	[5]
Augmented cube AQ_n	$K_{1,r}, P_\ell, C_m$	$1 \leq r \leq 6, 1 \leq \ell \leq 2n-1, 3 \leq m \leq 2n-1$	[24]
	$K_{1,r}$	$4 \leq r \leq \frac{3n-15}{4}$	[4]
Augmented k-ary n-cube $AQ_{n,k}$	P_ℓ	$n \geq 2, k \geq 3, 1 \leq \ell \leq 4n-2$	[7]
	$K_{1,r}, C_4$	$1 \leq r \leq 5$	[34]
Balanced hypercube BQ_n	$K_{1,r}, P_\ell, C_{6m}$	$r \in \{4,5\}, 4 \leq \ell \leq n, 6 \leq 6m \leq n$	[32]
	$K_{1,r}, P_\ell, C_4$	$1 \leq r \leq 2n, 1 \leq \ell \leq 7$	[55]
Crossed cube CQ_n	$K_{1,1}, K_{1,3}, P_\ell, C_4$	$3 \leq \ell \leq n$	[40]
Divide-and-swap cube DSC_n	$K_{1,r}, C_4$	$1 \leq r \leq \log_2 n$	[65]
Divide-and-swap k-ary n-cube DSC_n^k	$K_{1,r}$	$1 \leq r \leq \log_2 n + 2$	[47]
Exchanged hypercube $EH(s,t)$	$K_{1,r}, P_\ell, C_{2m}$	$r \leq s-1 \leq t-1, \ell \geq 2, 6 \leq 2m \leq s-1$	[33]
Folded cube FQ_n	$K_{1,3}, P_\ell, C_{2\ell}$	$n \geq 7, 2 \leq \ell \leq n$	[43]
	$K_{1,r}$	$n \geq 7, 2 \leq r \leq n-1$	[4]
Generalized hypercube $G(m_\ell, m_{\ell-1}, \ldots, m_1)$	$K_{1,r}, C_3, C_4, K_4$	$1 \leq r \leq \sum_{i=1}^{\ell-1} \frac{m_i}{\ell-1} - 1$	[45]
Hypercube-like networks HL_n	$K_{1,r}, C_4$	$n \geq 4, r \geq 1$	[30]
Twisted hypercubes TQ_n	$K_{1,r}, P_\ell$	$r \in \{3,4\}, 1 \leq \ell \leq n$	[26]
k-ary n-cube Q_n^k	$K_{1,r}$	$n \geq 2, k \geq 3, 1 \leq r \leq 3$	[35]
	P_ℓ, C_ℓ	$3 \leq \ell \leq 2n$	[36]

† $f(r)$ is a certain quadratic function, see [5]

The remaining parts of this paper are organized as follows. Section 2 provides some necessary definitions and preliminary results. Section 3 proves the correctness of Theorem 1. The last section concludes with a remark.

2 Preliminaries

For a positive integer n, let $\langle n\rangle = \{0, 1, \ldots, n-1\}$ and $[n] = \{1, 2, \ldots, n-1, n\}$. Let $G = (V(G), E(G))$ be a graph. We define the *neighborhood* and the *closed neighborhood* of a vertex $u \in V(G)$ by $N_G(u) = \{v \in V(G)\colon uv \in E(G)\}$ and $N_G[u] = N_G(u) \cup \{u\}$, respectively. Denote $d_G(u) = |N_G(u)|$ the *degree* of u in G and $\delta(G) = \min\{d_G(u)\colon u \in V(G)\}$ the minimum degree of G. For a subset $U \subseteq V(G)$, let $N_G(U) = \cup_{u\in U} N_G(u) \setminus U$, $N_G[U] = N_G(U) \cup U$, and $G[U]$ be the subgraph of G induced by U. For convenience, we use $[U]$ instead of $G[U]$, and omit the subscript G in notations $N_G(u)$, $N_G[u]$, $d_G(u)$, $N_G(U)$, and $N_G[U]$ if the graph G is evident from the context. The *distance* between u and v in G, denoted by $\mathrm{dist}_G(u, v)$, is the number of edges in a shortest path between u and v in G. Let $\mathcal{F}$ be a set of subgraphs of G and $F \in \mathcal{F}$. We use $G - F$ to denote $G[V(G) \setminus V(F)]$ and $G - \mathcal{F}$ to denote $G[V(G) \setminus \cup_{F\in\mathcal{F}} V(F)]$. For flexible use of notations, $G - F$ and $G - V(F)$ (resp. $G - \mathcal{F}$ and $G - V(\mathcal{F})$) represent the same meaning and can be used interchangeably. If a graph H is a star (i.e., $H \cong K_{1,r}$), we use $c(H)$ to denote the *center vertex* with degree r in H. For notations not explicitly explained here, we refer to [8].

Let $u = u_1u_0$ and $v = v_1v_0$ be two binary strings of length two. We say that u and v are *pair-related*, denoted by $u \sim v$ if $(u, v) \in \{(00, 00), (01, 11), (11, 01), (10, 10)\}$. Otherwise, denote $u \nsim v$ to mean u and v are not pair-related. Using this pair relation for binary strings, Efe [14] gave the following definition of crossed cubes.

Definition 1 (see [14]). The *n-dimensional crossed cube* CQ_n is defined by the following recursive fashion:
1. $CQ_1 \cong K_2$ has two vertices with labels 0 and 1.
2. For $n \geq 2$, CQ_n is composed of two subcubes CQ^0_{n-1} and CQ^1_{n-1} such that each subcube CQ^i_{n-1} for $i \in \{0, 1\}$ is isomorphic to CQ_{n-1} and every vertex in CQ^i_{n-1} is labeled by an n-bit binary string with the leading bit i. Two vertices $u = 0u_{n-2}\cdots u_1u_0 \in V(CQ^0_{n-1})$ and $v = 1v_{n-2}\cdots v_1v_0 \in V(CQ^1_{n-1})$ are adjacent if and only if
(a) $u_{n-2} = v_{n-2}$ if n is even, and
(b) $u_{2i+1}\,u_{2i} \sim v_{2i+1}\,v_{2i}$ for $0 \leq i < \lfloor\frac{n-1}{2}\rfloor$,
where u and v are called the $(n-1)$*-th neighbors* to each other, and denote as $v = u^{n-1}$ and $u = v^{n-1}$.

According to the above definition, the i-th neighbor of a vertex $u = u_{n-1}u_{n-2}\cdots u_1 u_0 \in V(CQ_n)$ for $i \in \langle n\rangle$ can be represented as follows:

$$u^i = \begin{cases} u_{n-1}\,u_{n-2}\;\cdots\;u_{i+1}\bar{u}_i(u_{i-1}\,u_{i-2})\;\cdots\;(u_1\,u_0), & \text{if } i \text{ is even;} \\ u_{n-1}\,u_{n-2}\;\cdots\;u_{i+1}\bar{u}_i\,u_{i-1}(u_{i-2}\,u_{i-3})\;\cdots\;(u_1\,u_0), & \text{if } i \text{ is odd,} \end{cases}$$

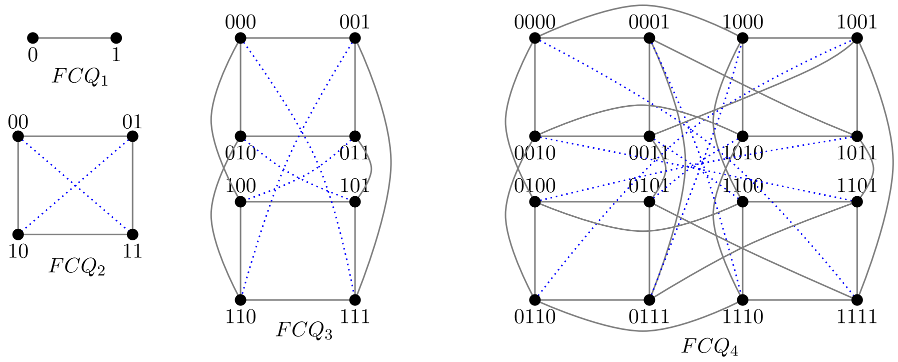


Fig. 1 FCQ_n for $n = 1, 2, 3, 4$, where dotted lines indicate complement edges

where $\bar{u}_i = 1 - u_i$ for $u_i \in \{0, 1\}$, and the notation $(u_j u_{j-1})$ denotes the pair-related operation on two consecutive bits u_j and u_{j-1}. Using this representation of neighbors, we can easily extend it to represent multiple levels of neighbor relationships. If v is the j-th neighbor of u^i, we denote $v = u^{i,j}$. Furthermore, if w is the k-th neighbor of v, we denoted $w = v^k = u^{i,j,k}$, and so on. Particularly, the $(n-1)$-th neighbor of u is called the *external neighbor* of u and is denoted by $\text{ex}(u) = u^{n-1}$.

Inspired by the idea of folded hypercube, Zhang [63] introduced folded crossed cube to strengthen the structure of CQ_n. For each $u = u_{n-1}u_{n-2}\cdots u_1u_0 \in V(CQ_n)$, let $\bar{u} = \overline{u_{n-1}u_{n-2}\cdots u_1u_0} = \bar{u}_{n-1}\bar{u}_{n-2}\cdots\bar{u}_1\bar{u}_0$. Then, u and $\bar{u}$ are called *complementary vertices*.

Definition 2 (see [63]). The *n-dimensional folded crossed cube FCQ_n* is constructed from CQ_n by adding a set of edges between complementary vertices. Formally, let $M = \{u\bar{u}\colon\ u \in V(CQ_n)\}$ be the set of *complementary edges*, where $\bar{u}$ is called the *complementary neighbor* of u and is denoted by $\text{co}(u) = \bar{u}$. Then, FCQ_n has the vertex set $V(FCQ_n) = V(CQ_n)$ and edge set $E(FCQ_n) = E(CQ_n) \cup M$.

FCQ_n have 2^n vertices and $(n+1)2^{n-1}$ edges. For any vertex $u \in V(CQ^i_{n-1})$ with $i \in \{0, 1\}$, it has only two neighbors in CQ^{1-i}_{n-1}, i.e., $\text{ex}(u)$ and $\text{co}(u)$. For $S \subseteq V(FCQ_n)$, abbreviate $N_{FCQ_n}(S)$ as $N(S)$. For any subgraph $\mathcal{S}$ of FCQ_n, abbreviate $N_{FCQ_n}(V(\mathcal{S}))$ as $N(\mathcal{S})$. The FCQ_n for $n \in [4]$ is shown in Fig. 1.

In what follows, we present some structure properties of FCQ_n.

Lemma 2.1 (see [9]). *For $n \geq 4$, FCQ_n has the following properties:*

(1) *FCQ_n contains no triangle.*

(2) *For two distinct vertices $u, v \in V(FCQ_n)$, $|N(u) \cap N(v)| \leq 2$.*

From Lemma 2.1, the following property directly holds.

Lemma 2.2 *Let $H \cong K_{1,r}$ be a subgraph of FCQ_n. If $u \in V(FCQ_n - H)$, then $|N(u) \cap V(H)| \leq 2$;*

Lemma 2.3 (see [3]). Let H be a subgraph of FCQ_n. For any edge $uv \in E(FCQ_n - H)$, if $H \cong K_{1,r}$ with $2 \leq r \leq n+1$, then $|N(\{u, v\}) \cap V(H)| \leq 3$.

Lemma 2.4 (see [21]). *For $n \geq 5$, let $u, v \in V(CQ^0_{n-1})$ in FCQ_n. If* $\mathrm{co}(u) = \mathrm{ex}(v)$*, then*

$$\begin{cases} \mathrm{dist}_{CQ^0_{n-1}}(u, v) = \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is odd;} \\ \lceil \frac{n}{2} \rceil \leq \mathrm{dist}_{CQ^0_{n-1}}(u, v) \leq \lceil \frac{n+1}{2} \rceil, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 2 (see [3]). *For $n \geq 8$ and $2 \leq r \leq \frac{n}{2}$, then $\kappa(FCQ_n; K_{1,r}) = \kappa^s(FCQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$.*

Theorem 3 (see [3]). *For $n \geq 6$ and $3 \leq k \leq 2n + 1$, then*

$$\kappa(FCQ_n; P_k) = \kappa^s(FCQ_n; P_k) = \begin{cases} \lceil \frac{2(n+1)}{k+1} \rceil, & \text{if } k \text{ is odd;} \\ \lceil \frac{2(n+1)}{k} \rceil, & \text{if } n \text{ is even.} \end{cases}$$

At the end of this section, we introduce two functions that play a crucial role in this paper. For $n \geq 4$, define

$$f(x) = -\frac{x^2}{2} + (n - \frac{1}{2})x + 1, \text{ where } 1 \leq x \leq n + 1, \quad (1)$$

and

$$g(x) = -\frac{x^2}{2} + (2n - \frac{3}{2})x + 2 - n^2, \text{ where } n + 2 \leq x \leq 2n. \quad (2)$$

Note that Eq. (1) firstly appeared in [18] to determine the t/k-diagnosability of BC_n, and Eq. (2) presented in [52] to solve the minimum neighborhood problem of BC_n. Yang and Lin [51] explored the extra connectivity of BC networks using Eqs. (1) and (2).

Lemma 2.5 (see [51, 53]). *Let BC_n be an n-dimensional bijective connection networks for $n \geq 4$, and $F \subseteq V(BC_n)$ be a faulty set of vertices. Then,*

(1) If $|F| < f(k)$ for $1 \leq k \leq n - 3$, then $BC_n - F$ contains exactly one large component with at least $2^n - |F| - (k - 1)$ vertices, i.e., the number of vertices in all small components is at most $k - 1$.

(2) If $|F| < f(k)$ for $n - 2 \leq k \leq n + 1$, then $BC_n - F$ contains exactly one large component with at least $2^n - |F| - (n + 1)$ vertices, i.e., the number of vertices in all small components is at most $n + 1$.

(3) If $|F| < g(k)$ for $n + 2 \leq k \leq 2n - 4$, then $BC_n - F$ contains exactly one large component with at least $2^n - |F| - (k - 1)$ vertices, i.e., the number of vertices in all small components is at most $k - 1$.

3 Proof of Theorem 1

Lemma 3.1 *If $n \geq 5$ and $3 \leq r \leq n$, then $\kappa(FCQ_n; K_{1,r}) \leq \lceil \frac{n+1}{2} \rceil$ and $\kappa^s(FCQ_n; K_{1,r}) \leq \lceil \frac{n+1}{2} \rceil$.*

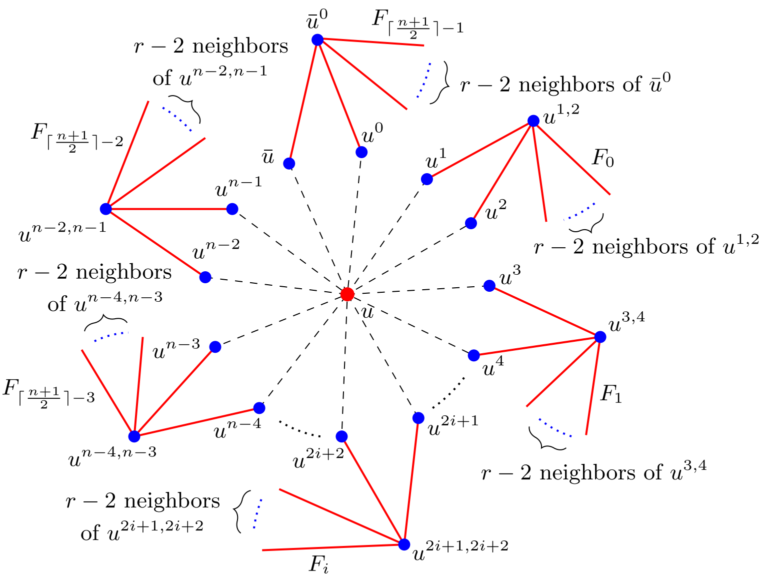


Fig. 2 Illustration of Lemma 3.1: A $K_{1,r}$-structure cut of FCQ_n when n is odd and $3 \le r \le n$

Proof First, we prove $\kappa(FCQ_n; K_{1,r}) \le \lceil\frac{n+1}{2}\rceil$. We will find a $K_{1,r}$-structure cut $\mathcal{F}$ with $|\mathcal{F}| = \lceil\frac{n+1}{2}\rceil$ such that $FCQ_n - \mathcal{F}$ is disconnected and it takes a singleton $u \in V(FCQ_n)$ as a trivial component. Clearly, $\lfloor\frac{n-1}{2}\rfloor = \lceil\frac{n+1}{2}\rceil - 1$ if n is odd, and $\lfloor\frac{n-1}{2}\rfloor = \lceil\frac{n+1}{2}\rceil - 2$ if n is even. For each $i \in \langle\lfloor\frac{n-1}{2}\rfloor\rangle$, let $I_i = \langle r\rangle\backslash\{2i+1, 2i+2\}$. Let $\mathcal{F} = \{F_0, F_1, \ldots, F_{\lceil\frac{n+1}{2}\rceil-2}, F_{\lceil\frac{n+1}{2}\rceil-1}\}$, where

$$V(F_i) = \{u^{2i+1}, u^{2i+1,\,2i+2}, u^{2i+2}\} \cup \{u^{2i+1,\,2i+2,\,j}:\ j \in I_i]\} \quad \text{for } i \in \langle\lfloor\frac{n-1}{2}\rfloor\rangle,$$

$$V(F_{\lceil\frac{n+1}{2}\rceil-1}) = \{\bar{u}, \bar{u}^0, u^0\} \cup \{\bar{u}^{0,j}:\ j \in [r-2]\},$$

and

$$V(F_{\lceil\frac{n+1}{2}\rceil-2}) = \{u^{n-1}, u^{n-1,\,1}\} \cup \{u^{n-1,\,1,\,j}:\ 2 \le j \le r\} \quad \text{if } n \text{ is even.}$$

Obviously, $F_i \cong K_{1,r}$ for each $F_i \in \mathcal{F}$. As $N(u) \subseteq \cup_{F_i\in\mathcal{F}} V(F_i)$, $FCQ_n - \mathcal{F}$ is disconnected, and $\mathcal{F}$ is a $K_{1,r}$-structure cut of FCQ_n. See Fig. 2 as an illustration for the description of a $K_{1,r}$-structure cut $\mathcal{F}$ when n is odd. Thus, $\kappa(FCQ_n; K_{1,r}) \le |\mathcal{F}| = \lceil\frac{n+1}{2}\rceil$. Since $\kappa^s(FCQ_n; K_{1,r}) \le \kappa(FCQ_n; K_{1,r})$, it implies that $\kappa^s(FCQ_n; K_{1,r}) \le \lceil\frac{n+1}{2}\rceil$. □

Note that FCQ_n contains two disjoint subgraphs CQ^0_{n-1} and CQ^1_{n-1} such that $CQ^0_{n-1} \cong CQ^1_{n-1} \cong CQ_{n-1}$. We will simply denote CQ^i_{n-1} as L_i for $i \in \{0, 1\}$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_t\}$, not necessarily pairwise non-isomorphic, be any $K_{1,r}$-substructure cut of FCQ_n for $3 \le r \le n$ such that every faulty star $F_i \in \mathcal{F}$ is isomorphic to a connected subgraph of $K_{1,r}$. For each $j \in \{0, 1\}$, let $\mathcal{F}^j$ be a set of subgraphs of L_j such that $V(\mathcal{F}^j) = V(\mathcal{F}) \cap V(L_j)$ and every elements of $\mathcal{F}^j$ is isomorphic to a subgraph of $K_{1,r}$. Let $C = \{c(F_i) : F_i \in \mathcal{F}\}$ be the set of all center vertices of faulty stars in $\mathcal{F}$. It is clear that $|C| = |\mathcal{F}|$ and $|V(\mathcal{F}^j)| \leq (r + 1) \cdot |C|$ for each $j \in \{0, 1\}$. Before giving the lower bounds of $\kappa(FCQ_n; K_{1,r})$ and $\kappa^s(FCQ_n; K_{1,r})$, we provide some properties related to center vertices of certain elements $F \in \mathcal{F}$ as follows.

Lemma 3.2 *For $n \geq 7$, let $z = 0z_{n-2}z_{n-3} \cdots z_1z_0 \in V(L_0)$, $x = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(\mathrm{co}(z^i))))$, and $y = \mathrm{co}(\mathrm{ex}(\mathrm{co}(\mathrm{ex}(z^i))))$ in FCQ_n. Then, x (resp. y) is a neighbor of z if and only if the following conditions hold:*

(1) $i \in \{1, n-3\}$ *for n even, or* $i \in \{1, n-2\}$ *for n odd;*

(2) $z_jz_{j-1} \in \{01, 11\}$ *where* $j = 2\ell + 1$ *for each* $\ell \in [\lceil \frac{n}{2} \rceil - 3]$*;*

(3) $z_1z_0 \in \{00, 10\}$.

Proof The proof of Lemma 3.2 is given in Appendix A. □

Lemma 3.3 *For $n \geq 5$, let $z = 0z_{n-2}z_{n-3} \cdots z_1z_0 \in V(L_0)$ and let $F \cong K_{1,r}$ and $F' \cong K_{1,r'}$ be two subgraphs of FCQ_n such that $c(F), c(F') \in V(L_0) \setminus \{z\}$. If $\mathrm{ex}(z) \in V(F)$ and $\mathrm{co}(z) \in V(F')$, then $c(F) \neq c(F')$.*

Proof The proof of Lemma 3.3 is given in Appendix B. □

Lemma 3.4 *For $n \geq 7$, let $z = 0z_{n-2}z_{n-3} \cdots z_1z_0 \in V(L_0)$ and $x, y \in N_{L_0}(z)$ be any two distinct neighbors of z. For each $t \in [2]$, let $F_t \cong K_{1,r_t}$ and $F'_t \cong K_{1,r'_t}$ be subgraphs of FCQ_n such that $c(F_t), c(F'_t) \in V(L_0)\backslash\{z\}$. Suppose $\mathrm{co}(x) \in V(F_1)$, $\mathrm{ex}(x) \in V(F_2)$, $\mathrm{co}(y) \in V(F'_1)$, and $\mathrm{ex}(y) \in V(F'_2)$. Then, the following two assertions hold:*

(1) *$c(F_1)$, $c(F_2)$, $c(F'_1)$, and $c(F'_2)$ are pairwise distinct if one of the following conditions holds:*

a. $z_1z_0 \notin \{00, 10\}$*;*

b. *There is an integer $j = 2\ell + 1$ for $\ell \in [\lceil \frac{n}{2} \rceil - 3]$ such that $z_jz_{j-1} \notin \{01, 11\}$;*

c. *$\{x, y\} \cap \{z^1, z^{n-3}\} = \emptyset$ for n even, or $\{x, y\} \cap \{z^1, z^{n-2}\} = \emptyset$ for n odd.*

(2) *Let $z_1z_0 \in \{00, 10\}$ and $z_jz_{j-1} \in \{01, 11\}$ where $j = 2\ell + 1$ for each $\ell \in [\lceil \frac{n}{2} \rceil - 3]$. If n is even and $\{x, y\} = \{z^1, z^{n-3}\}$ or n is odd and $\{x, y\} = \{z^1, z^{n-2}\}$ such that $c(F_1), c(F_2), c(F'_1), c(F'_2) \notin \{x, y\}$, then $c(F_1) = c(F'_2)$ and $c(F_2) = c(F'_1)$.*

Proof The proof of Lemma 3.4 is given in Appendix C. □

Next, we will prove that $|\mathcal{F}| \geq \lceil \frac{n+1}{2} \rceil$. Suppose on the contrary that $|\mathcal{F}| \leq \lceil \frac{n+1}{2} \rceil - 1$. Recall $C = \{c(F) \colon F \in \mathcal{F}\}$. Let $C^i = C \cap V(L_i)$ for $i \in \{0, 1\}$. Without loss of generality, assume $|C^1| = p \leq q = |C^0|$. We will prove that $FCQ_n - \mathcal{F}$ is connected through the following lemmas, which leads to a contradiction.

Lemma 3.5 *For $n \geq 9$, if $|\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$, then $L_1 - \mathcal{F}^1$ is connected.*

Proof As $|C^0| + |C^1| = |C| = |\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$ and $|C^1| = p \leq \lfloor |C|/2 \rfloor \leq \lfloor \lfloor \frac{n}{2} \rfloor /2 \rfloor = \lfloor \frac{n}{4} \rfloor$, it implies $|C^0| = q \leq \lfloor \frac{n}{2} \rfloor - p$. Suppose on the contrary that $L_1 - \mathcal{F}^1$ is disconnected. Let $\mathcal{H}_1$ be a smallest component of $L_1 - \mathcal{F}^1$. We prove the following assertion.

Claim A $|V(\mathcal{H}_1)| \le \frac{n+3}{2} - 1$.

Proof For any center vertex $v \in C^1$, one has that $d_{L_1}(v) \le n-1$. For any $F \in \mathcal{F}$ with $c(F) \in C^1$, let $F \cong K_{1,r_s}$. Obviously, $r_s \le n-1$. Since $\mathcal{F}^j$ is the set of subgraphs of L_j such that $V(\mathcal{F}^j) = V(\mathcal{F}) \cap V(L_j)$, it follows that

$$\begin{aligned}|V(\mathcal{F}^1)| &\le (r_s+1)\cdot|C^1| + \sum_{c(F_i)\in C^0} |V(F_i)\cap V(L_1)| \\ &\le n\cdot|C^1| + 2|C^0| \quad \text{(Note that every vertex in} \\ &\qquad\qquad\qquad\qquad L_0 \text{ has exactly two neighbors in } L_1.) \\ &\le n\cdot p + 2\cdot(\lfloor\frac{n}{2}\rfloor - p) \\ &\le (n-2)\cdot p + n \\ &\le (n-2)\cdot\frac{n}{4} + n \\ &\le \frac{n^2+2n}{4}.\end{aligned}$$

Let $|V(\mathcal{H}_1)| = m$. We now prove that $|V(\mathcal{F}^1)| < f(\frac{n+3}{2})$ for $n \ge 9$, where $f(\cdot)$ is defined in Eq. (1):

$$\begin{aligned}f\left(\frac{n+3}{2}\right) - |V(\mathcal{F}^1)| &\ge \left(-\frac{(\frac{n+3}{2})^2}{2} + (n-\frac{1}{2})(\frac{n+3}{2}) + 1\right) - \frac{n^2+2n}{4} \\ &= -\frac{n^2+6n+9}{8} + \frac{2n^2+5n+1}{4} - \frac{n^2+2n}{4} \\ &= \frac{n^2-7}{8} \\ &> 0.\end{aligned}$$

By Lemma 2.5(1), as $|V(\mathcal{F}^1)| < f(\frac{n+3}{2})$ and $\frac{n+3}{2} \le n-3$ for $n \ge 9$, the number of vertices in all small components of $L_1 - \mathcal{F}^1$ is at most $\frac{n+3}{2} - 1$. This implies that $|V(\mathcal{H}_1)| = m \le \frac{n+3}{2} - 1$. □

If $m = 1$, denote $V(\mathcal{H}_1) = \{u\}$. For any $F \in \mathcal{F}$, if $c(F) \in C^0$, by Lemma 2.4, one has that $|N_{L_1}(u) \cap V(F \cap L_1)| \le 1$; if $c(F) \in C^1$, by Lemma 2.2, then $|N_{L_1}(u) \cap V(F \cap L_1)| \le 2$. As a result,

$$\begin{aligned}|N_{L_1}(u) \cap V(\mathcal{F}^1)| &\le 2|C^1| + |C^0| \le 2p + (\lfloor\frac{n}{2}\rfloor - p) \\ &\le \lfloor\frac{n}{4}\rfloor + \lfloor\frac{n}{2}\rfloor < n-1 = |N_{L_1}(u)|,\end{aligned}$$

which contradicts that $\{u\}$ is a component of $L_1 - \mathcal{F}^1$. Thus $|V(\mathcal{H}_1)| = m \ge 2$.

Let u and v be two adjacent vertices in $\mathcal{H}_1$. Clearly, $|N_{\mathcal{H}_1}(\{u,v\})| \leq m-2$. Since $\mathcal{H}_1$ is a component of $L_1 - \mathcal{F}^1$, we have $N_{L_1}(\mathcal{H}_1) \subseteq V(\mathcal{F}^1)$. Then

$$\begin{aligned}|N_{L_1-\mathcal{H}_1}(u) \cup N_{L_1-\mathcal{H}_1}(v)| &= |N_{L_1}(u)| + |N_{L_1}(v)| - |N_{\mathcal{H}_1}(\{u,v\})| \\ &\quad -|\{u,v\}| \geq 2(n-1)-(m-2)-2.\end{aligned}$$

That is, u and v has at least $2n-m-2$ neighbors in $\mathcal{F}^1$. For any $F \in \mathcal{F}$, if $c(F) \in C^0$, by Lemma 2.4, one has that $|N_{L_1}(\{u,v\}) \cap V(F \cap L_1)| \leq 1$; if $c(F) \in C^1$, by Lemma 2.3, we have $|N_{L_1}(\{u,v\}) \cap V(F \cap L_1)| \leq 3$. So

$$2n-m-2 \leq 3|C^1| + |C^0| \leq 3p + (\lfloor \frac{n}{2} \rfloor - p) \leq 2 \cdot \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{2} \rfloor$$

When $n \geq 9$, this implies that

$$\begin{aligned}|V(\mathcal{H}_1)| = m &\geq (2n-2) - (2 \cdot \lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{2} \rfloor) \\ &\geq \begin{cases} n-2, & \text{if } n \text{ is even;} \\ n-3, & \text{if } n \text{ is odd,} \end{cases} \\ &> \frac{n+3}{2} - 1,\end{aligned}$$

which contradicts with Claim A. Thus, $L_1 - \mathcal{F}^1$ is connected. □

Lemma 3.6 *For $n \geq 9$, if $|\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$ and $|C^0| \geq |C^1| > 0$, then $|V(\mathcal{H}_0)| \geq n+2$, where $\mathcal{H}_0$ is one of the smallest component of $L_0 - \mathcal{F}^0$.*

Proof As $|C^0| + |C^1| = |C| = |\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$ and $|C^1| = p \geq 1$, it implies $|C^0| = q \leq \lfloor \frac{n}{2} \rfloor - 1$. We prove the following assertions.

Claim B $|C^1| \geq 2$.

Proof Suppose on the contrary that $|C^1| = p = 1$. Then, there exists an F $(\cong K_{1,r_s}) \in \mathcal{F}$ such that $c(F) \in C^1 \subseteq V(L_1)$. By Lemma 3.5, $L_1 - \mathcal{F}^1$ is connected. Thus, if $L_0 - \mathcal{F}^0$ is connected, then $L_0 - \mathcal{F}^0$ must be connected to $L_1 - \mathcal{F}^1$ as

$$(|V(L_1)| - |V(\mathcal{F}^1)|) - |V(\mathcal{F}^0)| = |V(L_1)| - |V(\mathcal{F})| \geq 2^{n-1} - (n+1)\lfloor \frac{n}{2} \rfloor > 0,$$

where the leftmost term means that the number of residual vertices after removing $\mathcal{F}^1$ from L_1 is more than the number of faulty stars with the center vertex located in L_0. Consequently, $FCQ_n - \mathcal{F}$ is connected, which contradicts with that $\mathcal{F}$ is a $K_{1,r}$-substructure cut of FCQ_n. Thus, we consider that $L_0 - \mathcal{F}^0$ is disconnected. Let $\mathcal{D}_0$ be any component of $L_0 - \mathcal{F}^0$. Note that $\mathcal{H}_0$ may or may not be $\mathcal{D}_0$.

Case B-1. $|V(\mathcal{D}_0)| = 1$.

Denote $\mathcal{D}_0 = \{w\}$. Since $\{w\}$ is a component of $L_0 - \mathcal{F}^0$, we have $N_{L_0}(w) \subseteq V(\mathcal{F}^0)$. We prove $|C^0| \geq \lfloor \frac{n}{2} \rfloor - 1$ by contradiction. Suppose not, and assume that

$|C^0| \leq \lfloor \frac{n}{2} \rfloor - 2$. For each $F \in \mathcal{F}$, by Lemma 2.2 and Lemma 2.4, we have $|N(w) \cap V(F)| \leq 2$ when $c(F) \in L_0$ and $|N(w) \cap V(F)| \leq 1$ when $c(F) \in L_1$. Thus,

$$|N_{L_0}(w) \cap V(\mathcal{F}^0)| \leq 2|C^0| + |C^1| = 2q + p \leq 2(\lfloor \frac{n}{2} \rfloor - 2) + 1 \leq n - 3 < n - 1 \\ = |N_{L_0}(w)|,$$

which contradicts with $N_{L_0}(w) \subseteq V(\mathcal{F}^0)$. Combining with $|C^0| \leq \lfloor \frac{n}{2} \rfloor - 1$, we have that $|C^0| = \lfloor \frac{n}{2} \rfloor - 1$.

Now, we prove that w is connected to $L_1 - \mathcal{F}^1$. On the contrary, we assume that $N_{L_1}(w) \subseteq V(\mathcal{F}^1)$. As $|C^1| = 1$, by Lemma 2.4, let $F_\ell \in \mathcal{F}$ with $c(F_\ell) \in C^1$ such that $|N_{L_1}(w) \cap V(F_\ell)| \leq 1$. As w has two neighbors in L_1, there is at least one neighbor of w that belongs to $\bigcup_{c(F) \in C^0} V(F \cap L_1)$ for $F \in \mathcal{F}$. Let $F_j \in \mathcal{F}$ be such a faulty star with $c(F_j) \in C^0$ such that $N_{L_1}(w) \cap V(F_j) \neq \emptyset$. By Lemma 2.4, $|N_{L_0}(w) \cap V(F_j)| = 0$. Then,

$$\begin{aligned} |N_{L_0}(w) \cap V(\mathcal{F}^0)| &\leq 2|C^0 \setminus \{c(F_j)\}| + |C^1| \\ &\leq 2(\lfloor \frac{n}{2} \rfloor - 2) + 1 < n - 1 = |N_{L_0}(w)|, \end{aligned}$$

which contradicts with that w is a component of $L_0 - \mathcal{F}^0$. Hence w is connected to $L_1 - \mathcal{F}^1$.

Case B-2. $|V(\mathcal{D}_0)| \geq 2$.

We prove that $\mathcal{D}_0$ is connected to $L_1 - \mathcal{F}^1$. Let $u \in V(\mathcal{D}_0)$ and $d_{\mathcal{D}_0}(u) = t$. Then $|N_{L_0}(u)| = n - 1 = |N_{\mathcal{D}_0}(u)| + |N_{\mathcal{F}^0}(u)|$. Since $|V(\mathcal{D}_0)| \geq 2$, we have $t \geq 1$. Let $Y = \bigcup_{u^i \in N_{\mathcal{D}_0}(u)} N_{L_1}(u^i)$. Clearly, if $Y \cap V(L_1 - \mathcal{F}^1) \neq \emptyset$, then $\mathcal{D}_0$ is connected to $L_1 - \mathcal{F}^1$. On the other hand, we suppose that $Y \subseteq V(\mathcal{F}^1)$. As $|C^1| = 1$, let $F_\ell \in \mathcal{F}$ with $c(F_\ell) \in C^1$. For any $u^i \in N_{\mathcal{D}_0}(u)$, by Lemma 2.4, $|N_{L_1}(u^i) \cap V(F_\ell)| \leq 1$. As u^i has two neighbors in L_1, there exists an $F_j \in \mathcal{F}$ with $c(F_j) \in C^0$ such that $|N_{L_1}(u^i) \cap V(F_j)| = 1$. Again, by Lemma 2.4, $|N_{L_0}(u) \cap V(F_j)| = 0$, i.e., F_j contains no neighbor of u in L_0. As $|N_{\mathcal{D}_0}(u)| = t$, at least t elements of $\mathcal{F}^0$ may contain no neighbor of u. Particularly, if $t \geq 3$, the number of elements in $\mathcal{F}^0$ that contain no neighbor of u may reduce to $t - 1$ caused by the following reason. For n even, if $u^1, u^{n-3} \in N_{\mathcal{D}_0}(u)$ and u fulfills the conditions of Lemma 3.4(2), then the center vertices of faulty stars containing $\mathrm{co}(u^1)$ and $\mathrm{ex}(u^{n-3})$ (resp. $\mathrm{ex}(u^1)$ and $\mathrm{co}(u^{n-3})$) in $\mathcal{F}^0$ are the same. Also, a similar argument by replacing u^{n-3} with u^{n-2} still holds for n odd. Thus,

$$|N_{\mathcal{F}^0}(u)| \leq \begin{cases} 2(|C^0| - t) + |C^1| \leq 2(\lfloor \frac{n}{2} \rfloor - 1 - t) + 1 \leq n - 2t - 1, & \text{if } t \in \{1, 2\}; \\ 2(|C^0| - (t-1)) + |C^1| \leq 2(\lfloor \frac{n}{2} \rfloor - t) + 1 \leq n - 2t + 1, & \text{if } 3 \leq t \leq n - 1. \end{cases}$$

As a result,

$$n - 1 = |N_{L_0}(u)| = |N_{\mathcal{F}^0}(u)| + |N_{\mathcal{D}_0}(u)| \leq \begin{cases} (n - 2t - 1) + t, & \text{if } t \in \{1, 2\}; \\ (n - 2t + 1) + t, & \text{if } 3 \leq t \leq n - 1 \end{cases} \\ < n - 1.$$

which is a contradiction. This means that $Y \nsubseteq V(\mathcal{F}^1)$. Hence, there exits a vertex $u^i \in N_{\mathcal{D}_0}(u)$ such that $\mathrm{co}(u^i)$ or $\mathrm{ex}(u^i)$ is located in $L_1 - \mathcal{F}^1$. By arbitrariness of u, $\mathcal{D}_0$ is connected to $L_1 - \mathcal{F}^1$.

Based on the discussions of Cases B-1 and B-2, if $L_0 - \mathcal{F}^0$ is disconnected and $|C^1| = 1$, then every component of $L_0 - \mathcal{F}^0$ is connected to $L_1 - \mathcal{F}^1$. As $L_1 - \mathcal{F}^1$ is connected by Lemma 3.5, it follows that $FCQ_n - \mathcal{F}$ is connected. This contradicts with the fact that $\mathcal{F}$ is a $K_{1,r}$-substructure cut of FCQ_n. Thus, $|C^1| \geq 2$. □

Claim C $|V(\mathcal{H}_0)| \geq 2$.

Proof Suppose on the contrary that $|V(\mathcal{H}_0)| = 1$. Denote $\mathcal{H}_0 = \{w\}$. Since $\mathcal{H}_0$ is a smallest component of $L_0 - \mathcal{F}^0$, we have $N_{L_0}(w) \subseteq V(\mathcal{F}^0)$. By Claim B, we have $|C^0| = |C| - |C^1| \leq \lfloor \frac{n}{2} \rfloor - 2$. One has that

$$
\begin{aligned}
|N_{L_0}(w) \cap V(\mathcal{F}^0)| &\leq 2|C^0| + |C^1| = |C| + |C^0| \\
&\leq \lfloor \frac{n}{2} \rfloor + (\lfloor \frac{n}{2} \rfloor - 2) < n - 1 = |N_{L_0}(w)|,
\end{aligned}
$$

which contradicts with $N_{L_0}(w) \subseteq V(\mathcal{F}^0)$. Thus $|V(\mathcal{H}_0)| \geq 2$. □

Let us now proceed to the proof of Lemma 3.6. Given a vertex $y \in V(\mathcal{H}_0)$, we let $N_1 = N_{\mathcal{H}_0}(y)$, $N_2 = N_{\mathcal{H}_0}(N_1) \setminus \{y\}$, $N_3 = N_{\mathcal{H}_0}(N_2) \setminus (N_1 \cup \{y\})$ and $N_4 = N_{\mathcal{H}_0}(N_3) \setminus (N_2 \cup N_1 \cup \{y\})$.

Case 1. $\lfloor \frac{n}{2} \rfloor \leq \delta(\mathcal{H}_0) \leq n - 1$.

Suppose $d_{\mathcal{H}_0}(y) = \delta(\mathcal{H}_0) = h \geq \lfloor \frac{n}{2} \rfloor$. Clearly, $|N_{\mathcal{H}_0}(w) \setminus \{y\}| \geq h - 1$ for any vertex $w \in N_1$. Next, we examine the minimum value of $|N_2|$. By Lemma 2.1(2), to achieve the minimum value of $|N_2|$, we need to maximize the number of vertices in N_1 that share common neighbors in N_2. Specifically, we aim to maximize the number of vertices in N_2 that share two common neighbors of y in N_1. We can obtain $|N_2| \geq h(h-1) - \binom{h}{2} = \frac{h(h-1)}{2}$. Thus,

$$
|V(\mathcal{H}_0)| \geq |\{y\}| + |N_1| + |N_2| \geq 1 + h + \frac{h(h-1)}{2} = \frac{h^2 + h + 2}{2}.
$$

As $h \geq \lfloor \frac{n}{2} \rfloor$, it follows $2h + 2 \geq n + 1$. When $n \geq 9$, one has that

$$
|V(\mathcal{H}_0)| - (n+1) \geq \frac{h^2 + h + 2}{2} - (2h + 2) = \frac{h^2 - 3h - 2}{2} > 0,
$$

where $h \geq \lfloor \frac{n}{2} \rfloor \geq 4$. Hence, $|V(\mathcal{H}_0)| \geq n + 2$.

Case 2. $\delta(\mathcal{H}_0) = 1$.

Suppose $d_{\mathcal{H}_0}(y) = \delta(\mathcal{H}_0) = 1$. Then, $|N_{\mathcal{F}^0}(y)| = |N_{L_0}(y) \setminus N_1| = (n-1) - 1 = n - 2$. By Claim B, we have $|C^0| \leq |C| - |C^1| \leq \lfloor \frac{n}{2} \rfloor - 2$. For any $F \in \mathcal{F}$, by Lemma 2.2 and Lemma 2.4, we have $|N_{L_0}(y) \cap V(F)| \leq 2$ when $c(F) \in C^0$ and $|N_{L_0}(y) \cap V(F)| \leq 1$ when $c(F) \in C^1$. Thus,

$$
n - 2 = |N_{\mathcal{F}^0}(y)| \leq 2|C^0| + |C^1| = |C| + |C^0| \leq \lfloor \frac{n}{2} \rfloor + (\lfloor \frac{n}{2} \rfloor - 2) \leq n - 2.
$$

Note that the above inequality implies that n is even and $|C^1| = 2$. Moreover, for any $F \in \mathcal{F}$, if $c(F) \in C^0$, we have $|N_{L_0}(y) \cap V(F)| = 2$; if $c(F) \in C^1$, then $|N_{L_0}(y) \cap V(F)| = 1$. As $d_{\mathcal{H}_0}(y) = |N_1| = 1$, we denote $N_{\mathcal{H}_0}(y) = \{w\}$. By Lemma 2.3, if $c(F) \in C^0$, then $|N_{L_0}(\{y, w\}) \cap V(F)| \leq 3$. It further implies $|N_{L_0}(w) \cap V(F)| \leq 1$. Also, by Lemmas 2.4 and 2.1(1), if $c(F) \in C^1$, we have $|N_{L_0}(w) \cap V(F)| = 0$. Thus, $|N_{\mathcal{F}^0}(w)| \leq |C^0| \leq \frac{n}{2} - 2$, which implies that

$$|N_2| = |N_{\mathcal{H}_0}(w) \setminus \{y\}| \geq |N_{L_0}(w)| - |\{y\}| - |N_{\mathcal{F}^0}(w)| \\ \geq (n-1) - 1 - (\frac{n}{2} - 2) = \frac{n}{2}. \tag{3}$$

As $n \geq 9$, $N_2 \neq \emptyset$. Let z be any vertex in N_2. For any $F \in \mathcal{F}$, if $c(F) \in C_0$, by Lemma 2.3, $|N_{L_0}(w, z) \cap V(F)| \leq 3$. As $|N_{L_0}(w) \cap V(F)| \leq 1$, it further implies $|N_{L_0}(z) \cap V(F)| \leq 2$. By Lemma 2.1(1), FCQ_n contains no triangle, and thus z and y are nonadjacent. By Lemma 2.1(2), y and z have at most two common neighbors in L_0. As $w \in N_{L_0}(y) \cap N_{L_0}(z)$, there is at most one faulty star $F' \in \mathcal{F}$ with $c(F') \in C^1$ such that $N_{L_0}(y) \cap V(F') = N_{L_0}(z) \cap V(F')$ and $|N_{L_0}(z) \cap V(F')| = 1$. Since $|C^1| = 2$, there is another faulty star $F'' \in \mathcal{F} \setminus \{F'\}$ with $c(F'') \in C^1$ such that $|N_{L_0}(y) \cap V(F'')| = 1$. Then, either $N_{L_0}(y) \cap V(F'') = \{\mathrm{ex}(c(F''))\}$ or $N_{L_0}(y) \cap V(F'') = \{\mathrm{co}(c(F''))\}$. By Lemma 2.4, $d_{L_0}(\mathrm{ex}(c(F'')), \mathrm{co}(c(F''))) \geq 5$ when $n \geq 9$, which implies that $\mathrm{ex}(c(F'')), \mathrm{co}(c(F'')) \notin N_{L_0}(z)$. Thus, $|N_{L_0}(z) \cap V(F')| + |N_{L_0}(z) \cap V(F'')| \leq 1$. Hence, we have

$$|N_{\mathcal{F}^0}(z)| = \sum_{c(F) \in C^0} |N_{L_0}(z) \cap V(F)| + \sum_{c(F) \in C^1} |N_{L_0}(z) \cap V(F)| \\ \leq 2|C^0| + 1 \leq 2(\frac{n}{2} - 2) + 1 \leq n - 3.$$

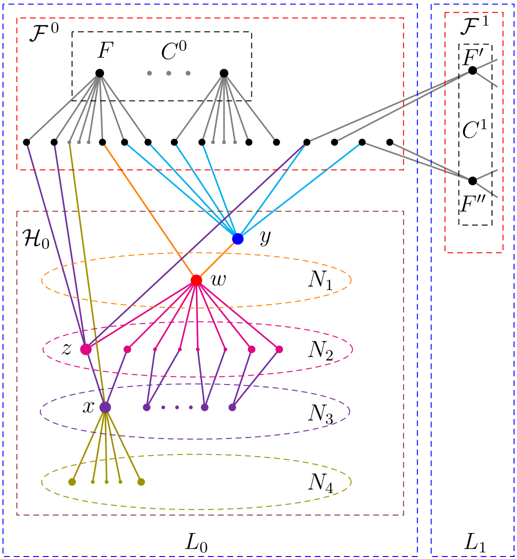
This implies that $|N_{\mathcal{H}_0}(z) \backslash (N_1 \cup N_2 \cup \{y\})| = |N_{L_0}(z)| - |\{w\}| - |N_{\mathcal{F}^0}(z)| \geq (n-1) - 1 - (n-3) = 1$, which means that every vertex $z \in N_2$ has at least one neighbor in N_3. However, by Lemma 2.1(2), at most two vertices of N_2 can share a common element in N_3 (see Fig. 3). By Eq. (3), $|N_2| \geq \frac{n}{2}$ and it follows that

$$|N_3| \geq \lceil \frac{|N_2|}{2} \rceil \geq \lceil \frac{n}{4} \rceil. \tag{4}$$

As $n \geq 9$, $N_3 \neq \emptyset$. Let x be any vertex in N_3. From above, we have $|N_{\mathcal{H}_0}(x) \cap (N_2 \cup \{w, y\})| \leq 2$. In particular, if $|N_3| = \lceil \frac{n}{4} \rceil$, by Lemma 2.3, any two vertices $x, x' \in N_3$ are nonadjacent (otherwise, a star centered at w with leaves at N_2 may intersect $N_{\mathcal{H}_0}(\{x, x'\})$ up to four vertices). Thus, $|N_{\mathcal{H}_0}(x) \cap (N_3 \cup N_2 \cup \{w, y\})| \leq 2$. For any $F \in \mathcal{F}$, if $c(F) \in C^0$, by Lemma 2.3, $|N_F(x)| \leq 1$; if $c(F) \in C^1$, by Lemma 2.1(1) and Lemma 2.4, $|N_{L_0}(x) \cap V(F)| = 0$. Thus,

$$|N_4| \geq |N_{\mathcal{H}_0}(x) \setminus (N_3 \cup N_2 \cup \{w, y\})| \\ = |N_{L_0}(x)| - |N_{N_2}(x)| - |N_{\mathcal{F}^0}(x)| \\ \geq (n-1) - 2 - |C^0|$$

Fig. 3 An illustration of Case 2 in Lemma 3.6



$$\geq n-3-(\frac{n}{2}-2)$$
$$=\frac{n}{2}-1. \quad (5)$$

By Eqs. (3)-(5), one has that

$$|V(\mathcal{H}_0)| \geq |\{y\}|+|N_1|+|N_2|+|N_3|+|N_4| \geq 1+1+\frac{n}{2}+\lceil\frac{n}{4}\rceil+\frac{n}{2}-1 > n+1.$$

Case 3. $2 \leq \delta(\mathcal{H}_0) \leq \lfloor\frac{n}{2}\rfloor - 1$.

Suppose $d_{\mathcal{H}_0}(y) = |N_1| = n_1$. Recall $|C^1| = p \geq 2$ and $|C^0| = q \leq \lfloor\frac{n}{2}\rfloor - 2$. For any $F \in \mathcal{F}$, by Lemma 2.2 and Lemma 2.4, we have $|N_{L_0}(y) \cap V(F)| \leq 2$ when $c(F) \in C^0$ and $|N_{L_0}(y) \cap V(F)| \leq 1$ when $c(F) \in C^1$. Let $\mathcal{B} = \{F \in \mathcal{F}\colon c(F) \in C^0, |N_{L_0}(y) \cap V(F)| = 2\}$ and $b = |\mathcal{B}|$. Clearly, $|N_{\mathcal{F}^0}(y)| \leq p+q+b$. As $|N_{L_0}(y)| = |N_{\mathcal{H}_0}(y)| + |N_{\mathcal{F}^0}(y)|$ and $|N_{\mathcal{H}_0}(y)| \geq \delta(\mathcal{H}_0) \geq 2$, we have

$$n_1 = |N_{\mathcal{H}_0}(y)| = |N_{L_0}(y)| - |N_{\mathcal{F}^0}(y)| \geq (n-1)-(p+q+b) \geq 2. \quad (6)$$

As $N_1 \neq \emptyset$, let $w \in N_1$ such that $|N_{\mathcal{H}_0}(w)| = \min\{|N_{\mathcal{H}_0}(v)|\colon v \in N_1\}$. For each $F \in \mathcal{F}$, by Lemma 2.3, $|N(\{y, w\}) \cap V(F)| \leq 3$. Thus, if $F \in \mathcal{B}$, then $|N_F(w)| \leq 1$; otherwise, $|N_F(w)| \leq 2$. As a result, $|N_{\mathcal{F}^0}(w)| \leq 2(q-b)+b$. Let $n_2 = |N_{\mathcal{H}_0}(w) \setminus \{y\}|$. Clearly,

$$n_2 = |N_{L_0}(w) \setminus \{y\}| - |N_{\mathcal{F}^0}(w)| \geq (n-2)-(2(q-b)+b). \quad (7)$$

Based on the consideration of a minority of neighbors on $\mathcal{H}_0$ in choosing w, every vertex of N_1 has at least n_2 neighbors in N_2. Consider two cases as follows.

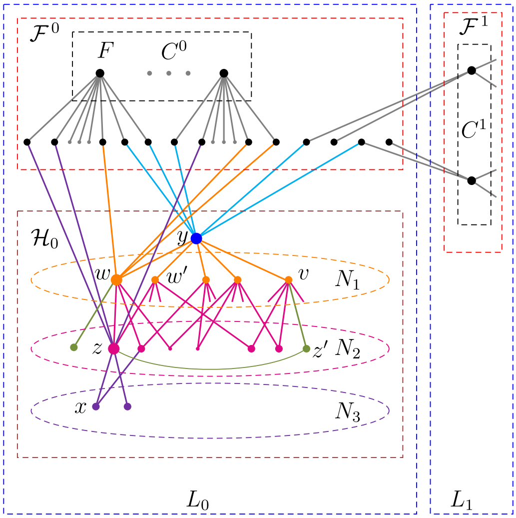


Fig. 4 An illustration of Case 3.1 in Lemma 3.6

Case 3.1. $n_2 \geq n_1$.

By Lemma 2.1(2) and $n_2 \geq n_1$, any two vertices of N_1 can share at most one common neighbor in N_2. This implies that

$$|N_2| \geq n_1 n_2 - \binom{n_1}{2} = n_1 n_2 - \frac{1}{2}(n_1^2 - n_1) \tag{8}$$

with equality holding in the situation where any two vertices of N_1 share two common neighbors in $\mathcal{H}_0$. In what follows, we show that the tight lower bound of $|V(\mathcal{H}_0)| \geq n+2$ can be attained in this specific situation. Overall, we can prove $|V(\mathcal{H}_0)| \geq n+2$ by induction.

As $q \leq \lfloor \frac{n}{2} \rfloor - 2$ and $0 \leq b \leq q$, one has that $n_2 \geq 2$. Thus, $N_2 \neq \emptyset$. Let z be a vertex in N_2 such that $N_{\mathcal{H}_0}(z) \cap N_1 = \{w, w'\}$. For each vertex $v \in N_1 \setminus \{w, w'\}$, by Lemma 2.3, there is at most one neighbor $z' \in N_{\mathcal{H}_0}(v) \cap N_2$ such that it is adjacent to z (otherwise, if two vertices $z', z'' \in N_{\mathcal{H}_0}(v) \cap N_2$ are adjacent to z, then a star centered at z with leaves including w, w', z', z'' may intersect $N_{\mathcal{H}_0}(\{y, v\})$ up to four vertices, see Fig. 4). This shows that $|N_{\mathcal{H}_0}(z) \cap N_2| \leq n_1 - 2$. Next, we consider the number of neighbors of z in $\mathcal{F}^0$. For each $F \in \mathcal{F}$, if $c(F) \in C^1$, by Lemma 2.1(1) and Lemma 2.4, $|N_{L_0}(z) \cap V(F)| = 0$. In addition, if $c(F) \in C^0$, by Lemma 2.2, $|N_{L_0}(z) \cap V(F)| \leq 2$. Moreover, since $|N_{L_0}(\{y, w\}) \cap V(F)| \leq 3$, if $F \notin \mathcal{B}$, then $|N_{L_0}(w) \cap V(F)|$ can be up to two, and it further implies that $|N_{L_0}(z) \cap V(F)| \leq 1$. Thus, $|N_{\mathcal{F}^0}(z)| \leq q + b$ (see Fig. 4). Then,

$$\begin{aligned} |N_3| &\geq |N_{\mathcal{H}_0}(z) \setminus (N_2 \cup N_1 \cup \{y\}))| \\ &= |N_{L_0}(z)| - |N_{N_1}(z)| - |N_{N_2}(z)| - |N_{\mathcal{F}^0}(z)| \\ &\geq (n-1) - 2 - (n_1 - 2) - (q + b). \end{aligned} \tag{9}$$

Hence, by Eqs. (8) and (9), one has that

$$\begin{aligned}
&|V(\mathcal{H}_0)| - (n+1)\\
&\quad \geq |\{y\}| + |N_1| + |N_2| + |N_3| - (n+1)\\
&\quad \geq 1 + n_1 + (n_1 n_2 - \frac{n_1^2}{2} + \frac{n_1}{2}) + (n-1) - 2 - (n_1 - 2) - (q+b) - (n+1)\\
&\quad \geq n_1 n_2 - \frac{n_1^2}{2} + \frac{n_1}{2} - (q+b+1). \qquad (10)
\end{aligned}$$

According to the above inequality, we define $f(n_1, n_2) = n_1 n_2 - \frac{1}{2}(n_1^2 - n_1) - (q + b + 1)$ to be a function of two variables. Clearly, we have the two partial derivatives $f_{n_1}(n_1, n_2) = n_2 - n_1 + \frac{1}{2}$ and $f_{n_2}(n_1, n_2) = n_1$. In this case, $f_{n_1}(n_1, n_2) > 0$ and $f_{n_2}(n_1, n_2) > 0$. As $n_2 \geq n_1$ and by Eq. (6), $n_1 \geq 2$, we claim that $f(n_1, n_2) > 0$ for all $n_1 \in \{2, 3, \ldots, n_2\}$. The proof is by induction on n_1.

When $n_1 = 2$, by Eq. (6), we have $(n-1) - (p+q+b) = 2$. Then, combining $2 \leq p \leq q$, $p+q \leq \lfloor \frac{n}{2} \rfloor$ and $b \leq q$, it follows that either $p = 2, q = \lfloor \frac{n}{2} \rfloor - 2, b = \lfloor \frac{n}{2} \rfloor - 3$ or $p = 3, q = b = \lfloor \frac{n}{2} \rfloor - 3$. By Eq. (7), we have $n_2 \geq (n-2) - (2q - b) \geq \lceil \frac{n}{2} \rceil - 1$. As $f(\cdot)$ is an increasing function with respect to both n_1 and n_2, one has that

$$\begin{aligned}
f(n_1, n_2) \geq f(2, n_2) \geq f(2, \lceil \frac{n}{2} \rceil - 1) &= 2 \cdot (\lceil \frac{n}{2} \rceil - 1) - 1 - (q+b+1)\\
&= (n-1) - (p+q+b) + p - 3 > 0.
\end{aligned}$$

Assume $f(n_1, n_2) > 0$ for $2 \leq n_1 \leq n_2 - 1$. That is,

$$f(n_1, n_2) = n_1 \cdot n_2 - \frac{n_1^2}{2} + \frac{n_1}{2} - (q+b+1) > 0.$$

We now consider $f(n_1 + 1, n_2)$ and, by induction hypothesis, we obtain

$$\begin{aligned}
f(n_1+1, n_2) &= (n_1+1) \cdot n_2 - \frac{(n_1+1)^2}{2} + \frac{n_1+1}{2} - (q+b+1)\\
&= \left(n_1 \cdot n_2 - \frac{n_1^2}{2} + \frac{n_1}{2} - (q+b+1)\right) + (n_2 - n_1) > 0.
\end{aligned}$$

Thus, $f(n_2, n_2) > 0$ and $|V(\mathcal{H}_0)| \geq f(n_2, n_2) + (n+1) > n+1$.

To help readers understand the above proof, we provide the analytic framework for this case in Fig. 5(a), which can also be conveniently compared with the following case.

Case 3.2. $n_2 < n_1$.

Recall that $w \in N_1$ is a vertex such that $|N_{\mathcal{H}_0}(w)| = \min\{|N_{\mathcal{H}_0}(v)| : v \in N_1\}$. As $|N_1| = n_1 > n_2$, let N_1' be a proper subset of N_1 and $n_1' = |N_1'|$ such that $n_1' \leq n_2$ and $w \in N_1'$. Then, $|N_1 \setminus N_1'| = n_1 - n_1' \geq n_1 - n_2 > 0$. Let $N_2' = N_{\mathcal{H}_0}(N_1') \setminus \{y\}$ and $N_3' = N_{\mathcal{H}_0}(N_2') \setminus (N_1' \cup \{y\})$. As $n_2 \geq n_1'$, arguments similar to Eqs. (8) and (9) in

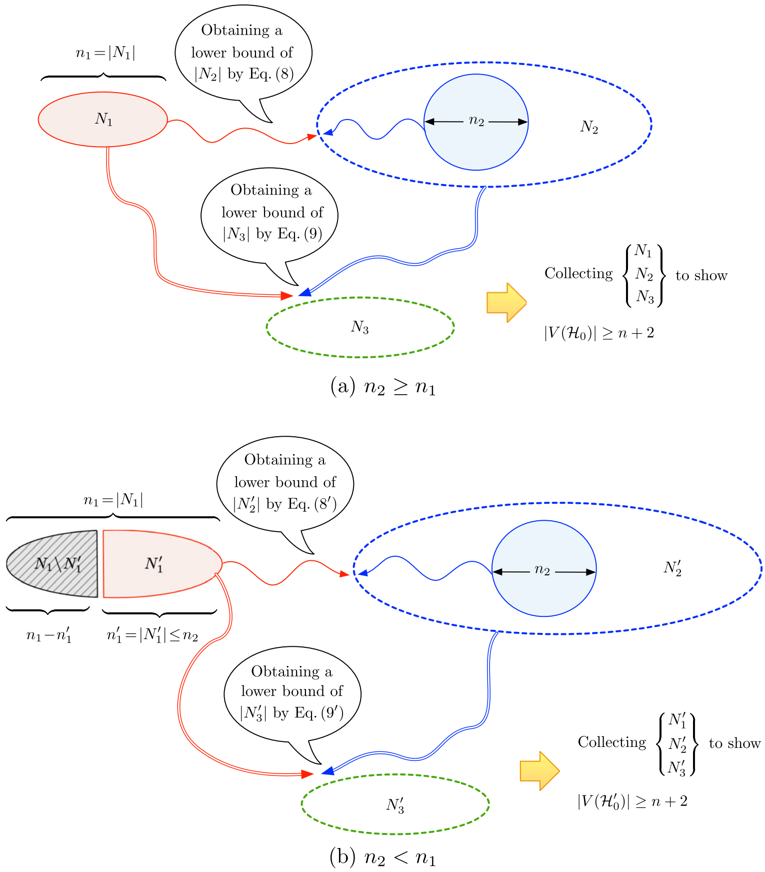


Fig. 5 An illustration of Case 3.2 in Lemma 3.6. (a) Case 3.1: $n_2 \geq n_1$; (b) Case 3.2: $n_2 < n_1$

Case 3.1 can estimate two lower bounds of $|N_2'|$ and $|N_3'|$ as follows.

$$|N_2'| \geq n_1' n_2 - \binom{n_1'}{2} = n_1' n_2 - \frac{1}{2}({n_1'}^2 - n_1') \tag{8'}$$

and

$$\begin{aligned} |N_3'| \geq & \quad |N_{\mathcal{H}_0}(z) \setminus (N_2' \cup N_1' \cup \{y\})| \\ = & \quad |N_{L_0}(z)| - |N_{N_1'}(z)| - |N_{N_2'}(z)| - |N_{\mathcal{F}^0}(z)|, \end{aligned} \tag{9'}$$

where $z \in N_2'$ has exactly two neighbors (including w) that are contained in N_1'. Let $\mathcal{H}_0' = \{y\} \cup N_1' \cup N_2' \cup N_3'$. By Eqs. (8') and (9'), a reasoning similar to Eq. (10) that follows a proof of induction can show that $|V(\mathcal{H}_0')| > n + 1$. Therefore, $|V(\mathcal{H}_0)| \geq |N_1 \backslash N_1'| + |V(\mathcal{H}_0')| \geq n + 2$ (see Fig. 5(b)).

Based on discussions of Cases 1–3, for any case of $1 \leq \delta(\mathcal{H}_0) \leq n-1$, if $n \geq 9$ and $C^j \neq \emptyset$ for $j \in \{0, 1\}$, then $|V(\mathcal{H}_0)| \geq n+2$. □

Proof of Theorem 1. By Lemma 3.1, it suffices to prove that $\kappa^s(FCQ_n; K_{1,r}) \geq \lceil \frac{n+1}{2} \rceil$. Suppose on the contrary that $\kappa^s(FCQ_n; K_{1,r}) \leq \lceil \frac{n+1}{2} \rceil - 1 = \lfloor \frac{n}{2} \rfloor$ and $\mathcal{F}$ is a $K_{1,r}$-substructure cut of FCQ_n with $|\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$. In what follows, we consider situations according to where center vertices of C are located.

Case 1. $C \subseteq V(L_0)$ or $C \subseteq V(L_1)$.

Without loss of generality, assume $C \subseteq V(L_0)$. Then, $|C^0| = |\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$. By Lemma 3.5, $L_1 - \mathcal{F}^1$ is connected. Thus, if a vertex $v \in V(L_0 - \mathcal{F}^0)$ with $|N_{L_1}(v) \cap V(L_1 - \mathcal{F}^1)| \geq 1$, then v is connected to $L_1 - \mathcal{F}^1$. As $\mathcal{F}$ is a $K_{1,r}$-substructure cut of FCQ_n, there is at least one vertex $z \in V(L_0 - \mathcal{F}^0)$ such that $|N_{L_1}(z) \cap V(L_1 - \mathcal{F}^1)| = 0$ and it is separated apart from $L_1 - \mathcal{F}^1$ in $FCQ_n - \mathcal{F}$. In this case, $\mathrm{ex}(z), \mathrm{co}(z) \in V(\mathcal{F}^1)$. Let S_1, S_2 be two faulty stars in $\mathcal{F}$ such that $\mathrm{ex}(z) \in V(S_1)$ and $\mathrm{co}(z) \in V(S_2)$. Clearly, $c(S_1), c(S_2) \in C^0$. By Lemma 3.3, one has that $c(S_1) \neq c(S_2)$. For each $i \in [2]$, by Lemma 2.4, we have $\mathrm{dist}_{L_0}(z, c(S_i)) \geq \lceil \frac{n}{2} \rceil \geq 5$ when $n \geq 9$, which further implies that $N_{L_0}(z) \cap V(S_i) = \emptyset$. Let $s = |N_{L_0 - \mathcal{F}^0}(z)|$. By Lemma 2.2,

$$|N_{\mathcal{F}^0}(z)| \leq 2(|C^0| - 2) \leq 2(\lfloor \frac{n}{2} \rfloor - 2) \leq n - 4,$$

and $s \geq d_{L_0}(z) - |N_{\mathcal{F}^0}(z)| \geq (n-1) - (n-4) = 3$. For each vertex $q \in N_{L_0 - \mathcal{F}^0}(z)$, its neighbors $\mathrm{ex}(q)$ and $\mathrm{co}(q)$ must be contained in $\mathcal{F}^1$. Otherwise, z is connects to $L_1 - \mathcal{F}^1$ through q and one of $\mathrm{ex}(q)$ and $\mathrm{co}(q)$, a contradiction (see Fig. 6). Let $x, y \in N_{L_0 - \mathcal{F}^0}(z)$ and F_1, F_1', F_2, F_2' be four stars in $\mathcal{F}$ such that $\mathrm{ex}(x) \in V(F_1)$, $\mathrm{co}(x) \in V(F_1')$, $\mathrm{ex}(y) \in V(F_2)$ and $\mathrm{co}(y) \in V(F_2')$. By Lemma 3.4, either $c(F_1), c(F_1'), c(F_2), c(F_2')$ are all distinct or $c(F_1) = c(F_2')$ and $c(F_2) = c(F_1')$. For each $i \in [2]$, by Lemma 2.4, we have $|N_{L_0}(z) \cap V(F_i)| = 0$ and $|N_{L_0}(z) \cap V(F_i')| = 0$ when $n \geq 9$. As x and y are arbitrary selected from $N_{L_0 - \mathcal{F}^0}(z)$, it results in 2 or 4 stars in $\mathcal{F}$ without intersection with $N_{L_0}(z)$, which means that for every additional vertex in $N_{L_0 - \mathcal{F}^0}(z)$, there will be one more star in $\mathcal{F}$ that has no intersection with $N_{L_0}(z)$. As $s \geq 3$, it derives that

$$|N_{L_0}(z) \cap V(\mathcal{F}^0)| \leq 2(\lfloor \frac{n}{2} \rfloor - s) \leq n - 2s.$$

As a result, for $n \geq 9$,

$$\begin{aligned} n - 1 = |N_{L_0}(z)| &= |N_{L_0}(z) \cap V(\mathcal{F}^0)| + |N_{L_0 - \mathcal{F}^0}(z)| \\ &\leq (n - 2s) + s = n - s < n - 2, \end{aligned}$$

a contradiction. Thus, there does not exist a vertex $z \in V(L_0 - \mathcal{F}^0)$ such that $|N_{L_1}(z) \cap V(L_1 - \mathcal{F}^1)| = 0$ and $FCQ_n - \mathcal{F}$ is connected. This again leads to a contradiction.

Case 2. $C \cap V(L_j) \neq \emptyset$ for each $j \in \{0, 1\}$.

Recall that Claim B of Lemma 3.6 shows that $|C^1| \geq 2$. As $|C| = |\mathcal{F}| \leq \lfloor \frac{n}{2} \rfloor$, we have $|C^0| = |C| - |C^1| \leq \lfloor \frac{n}{2} \rfloor - 2$. Clearly, if a faulty star $F \cong K_{1,r}$ with all leaves in L_0, then $r \leq n - 1$. Also, every faulty star with center in C^1 has at most two leaves

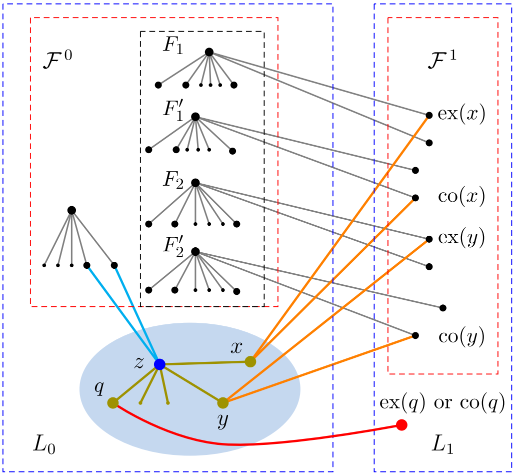


Fig. 6 An illustration of Case 1 in Theorem 1.1

in $\mathcal{F}^0$. It follows that

$$\begin{aligned}|V(\mathcal{F}^0)| &\leq (r+1)\cdot|C^0| + 2|C^1| \\ &= (r-1)\cdot|C^0| + 2|C| \\ &\leq (n-2)(\lfloor\frac{n}{2}\rfloor - 2) + 2\lfloor\frac{n}{2}\rfloor \\ &\leq \frac{n^2-4n+8}{2}.\end{aligned}$$

We now prove that $|V(\mathcal{F}^0)| < g(n+2)$ for $n \geq 9$, where $g(\cdot)$ is defined as in Eq. (2):

$$\begin{aligned}g(n+2) - |V(\mathcal{F}^0)| &\geq \left(-\frac{(n+2)^2}{2} + (2n-\frac{3}{2})(n+2) + 2 - n^2\right) - \frac{n^2-4n+8}{2} \\ &= \frac{5n-14}{2} > 0.\end{aligned}$$

By Lemma 2.5(3), the number of vertices in all small components of $L_0 - \mathcal{F}^0$ is at most $n+1$, which contradicts with Lemma 3.6 that the smallest component of $L_0-\mathcal{F}^0$ has size at least $n+2$. So $L_0 - \mathcal{F}^0$ is connected.

In addition, by Lemma 3.5, $L_1 - \mathcal{F}^1$ is connected. We will show that $L_1 - \mathcal{F}^1$ and $L_0 - \mathcal{F}^0$ are connected to each other through the following verification. For $n \geq 9$, we have

$$|V(L_1)| - |V(\mathcal{F}^1)| - |V(\mathcal{F}^0)| = |V(L_1)| - |V(\mathcal{F})| \geq 2^{n-1} - (n+1)\lfloor\frac{n}{2}\rfloor > 0.$$

This contradicts with the fact that $\mathcal{F}$ is a $K_{1,r}$-substructure cut of FCQ_n. Hence, $|\mathcal{F}| \geq \lfloor\frac{n}{2}\rfloor + 1 = \lceil\frac{n+1}{2}\rceil$. As the arbitrariness of $\mathcal{F}$, we have $\kappa(FCQ_n; K_{1,r}) \geq \kappa^s(FCQ_n; K_{1,r}) \geq \lceil\frac{n+1}{2}\rceil$.

By Lemma 3.1, $\kappa^s(FCQ_n; K_{1,r}) \leq \kappa(FCQ_n; K_{1,r}) \leq \lceil \frac{n+1}{2} \rceil$. Thus, we attain the desired result $\kappa^s(FCQ_n; K_{1,r}) = \kappa(FCQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$. □

4 Conclusion

In [3], Ba first studied the $K_{1,r}$-(sub)structure connectivity of n-dimensional folded crossed cube FCQ_n in her dissertation and obtained $\kappa(FCQ_n; K_{1,r}) = \kappa^s(FCQ_n; K_{1,r}) = \lceil \frac{n+1}{2} \rceil$ for $n \geq 8$ and $2 \leq r \leq \frac{n}{2}$ (see Theorem 2). Moreover, the P_k-(sub)structure connectivity of FCQ_n was also investigated for $n \geq 6$ and $3 \leq k \leq 2n+1$ (see Theorem 3). Note that if $k = 3$, the latter result is consistent with the previous one because $K_{1,2} \cong P_3$. Therefore, combined with the results of Theorem 3, we can extend our results for Theorem 1 to meet the overall range of $2 \leq r \leq n$.

Appendix A: Proof of Lemma 3.2

Let $n \geq 7$. Given a vertex $z = 0z_{n-2}z_{n-3}\cdots z_1z_0 \in V(L_0)$, we let

$$x = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(\mathrm{co}(z^i)))) = x_{n-1}x_{n-2}\cdots x_1x_0$$

and

$$y = \mathrm{co}(\mathrm{ex}(\mathrm{co}(\mathrm{ex}(z^i)))) = y_{n-1}y_{n-2}\cdots y_1y_0.$$

Recall that two binary strings of length two are pair-related provided they fulfill the relations:

$$00 \sim 00, \quad 01 \sim 11, \quad 11 \sim 01, \text{ and } 10 \sim 10.$$

Also, (z_jz_{j-1}) denotes the pair-related operation on two consecutive bits z_j and z_{j-1}. We consider the following two cases depending on the parity of n.

Case 1: n is even.

Case 1.1: If i is odd, we can derive the representation of x through the following:

$$\begin{aligned}
z^i &= 0\, z_{n-2}\; z_{n-3}\; z_{n-4} \cdots z_{i+2}\; z_{i+1}\; \bar{z}_i\; z_{i-1}\; (z_{i-2}\; z_{i-3}) \cdots (z_1\; z_0);\\
\mathrm{co}(z^i) &= 1\, \bar{z}_{n-2}\; \bar{z}_{n-3}\; \bar{z}_{n-4} \cdots \bar{z}_{i+2}\; \bar{z}_{i+1}\; z_i\; \bar{z}_{i-1}\; \overline{(z_{i-2}\; z_{i-3})} \cdots \overline{(z_1\; z_0)};\\
\mathrm{ex}(\mathrm{co}(z^i)) &= 0\, \bar{z}_{n-2}\, (\bar{z}_{n-3}\bar{z}_{n-4}) \cdots (\bar{z}_{i+2}\bar{z}_{i+1})\, (z_i\bar{z}_{i-1})\, (\overline{(z_{i-2}\; z_{i-3})}) \cdots (\overline{(z_1\; z_0)});\\
\mathrm{co}(\mathrm{ex}(\mathrm{co}(z^i))) &= 1\, z_{n-2}\, \overline{(\bar{z}_{n-3}\bar{z}_{n-4})} \cdots \overline{(\bar{z}_{i+2}\bar{z}_{i+1})}\, \overline{(z_i\bar{z}_{i-1})}\, \overline{(\overline{(z_{i-2}\; z_{i-3})})} \cdots \overline{(\overline{(z_1\; z_0)})};\\
x = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(\mathrm{co}(z^i)))) &= 0\, z_{n-2}(\overline{(\bar{z}_{n-3}\bar{z}_{n-4})}) \cdots (\overline{(\bar{z}_{i+2}\bar{z}_{i+1})})(\overline{(z_i\bar{z}_{i-1})})(\overline{(\overline{(\overline{(z_{i-2}\; z_{i-3})})})}) \cdots (\overline{(\overline{(\overline{(z_1\; z_0)})})}).
\end{aligned}$$

For $j \in \{1, 3, \ldots, i-2\}$, we check x_jx_{j-1} corresponding to $z_jz_{j-1} \in \{00, 01, 11, 10\}$ as follows:

$$x_jx_{j-1} = (\overline{(\overline{(z_jz_{j-1})})})$$

$$
= \begin{cases}
(\overline{(\overline{(00)})}) = (\overline{(\overline{00})}) = (\overline{(11)}) = (\overline{01}) = (10) = 10 & \text{if } z_j z_{j-1} = 00; \\
(\overline{(\overline{(01)})}) = (\overline{(\overline{11})}) = (\overline{(00)}) = (\overline{00}) = (11) = 01 & \text{if } z_j z_{j-1} = 01; \\
(\overline{(\overline{(11)})}) = (\overline{(\overline{01})}) = (\overline{(10)}) = (\overline{10}) = (01) = 11 & \text{if } z_j z_{j-1} = 11; \\
(\overline{(\overline{(10)})}) = (\overline{(\overline{10})}) = (\overline{(01)}) = (\overline{11}) = (00) = 00 & \text{if } z_j z_{j-1} = 10.
\end{cases}
\tag{A.1}
$$

As shown above, $x_j x_{j-1} = z_j z_{j-1}$ only when $z_j z_{j-1} \in \{01, 11\}$ and $x_j x_{j-1} \nsim z_j z_{j-1}$ for all $j \in \{1, 3, \ldots, i-2\}$. In addition, we check whether $x_i x_{i-1}$ and $z_i z_{i-1}$ are pair-related as follows:

$$
x_i x_{i-1} = (\overline{(z_i \bar{z}_{i-1})}) = \begin{cases}
(\overline{(0\bar{0})}) = (\overline{(01)}) = (\overline{11}) = (00) = 00 & \text{if } z_i z_{i-1} = 00; \\
(\overline{(0\bar{1})}) = (\overline{(00)}) = (\overline{00}) = (11) = 01 & \text{if } z_i z_{i-1} = 01; \\
(\overline{(1\bar{1})}) = (\overline{(10)}) = (\overline{10}) = (01) = 11 & \text{if } z_i z_{i-1} = 11; \\
(\overline{(1\bar{0})}) = (\overline{(11)}) = (\overline{01}) = (10) = 10 & \text{if } z_i z_{i-1} = 10.
\end{cases}
$$

Clearly, $x_i x_{i-1} = z_i z_{i-1}$ for all cases, and $x_i x_{i-1} \sim z_i z_{i-1}$ appears only when $z_i z_{i-1} \in \{00, 10\}$. Moreover, we check $x_j x_{j-1}$ corresponding to $z_j z_{j-1} \in \{00, 01, 11, 10\}$ for $j \in \{i+2, i+4, \ldots, n-3\}$ as follows:

$$
x_j x_{j-1} = (\overline{(\bar{z}_j \bar{z}_{j-1})}) = \begin{cases}
(\overline{(\bar{0}\bar{0})}) = (\overline{(11)}) = (\overline{01}) = (10) = 10 & \text{if } z_j z_{j-1} = 00; \\
(\overline{(\bar{0}\bar{1})}) = (\overline{(10)}) = (\overline{10}) = (01) = 11 & \text{if } z_j z_{j-1} = 01; \\
(\overline{(\bar{1}\bar{1})}) = (\overline{(00)}) = (\overline{00}) = (11) = 01 & \text{if } z_j z_{j-1} = 11; \\
(\overline{(\bar{1}\bar{0})}) = (\overline{(01)}) = (\overline{11}) = (00) = 00 & \text{if } z_j z_{j-1} = 10.
\end{cases}
\tag{A.2}
$$

Note that $x_j x_{j-1} \sim z_j z_{j-1}$ when $z_j z_{j-1} \in \{01, 11\}$ for all $j \in \{i+2, i+4, \ldots, n-3\}$. Particularly, if $\{i+2, i+4, \ldots, n-3\} \neq \emptyset$ (i.e., $i \leq n-5$), the highest bit where x differs from z is at position $n-3$ and $x_{n-4} = z_{n-4}$. By Eq. (A.1), $x_j x_{j-1} \nsim z_j z_{j-1}$ for all $j \in \{1, 3, \ldots, i-2\}$. Thus, if x is a neighbor of z such that $i \leq n-5$, then it must be the $(n-3)$-th neighbor and $i = 1$, in which $z_j z_{j-1} \in \{01, 11\}$ for all $j \in \{3, 5, \ldots, n-5\}$ and $z_1 z_0 \in \{00, 10\}$. Next, we consider $i = n-3$. In this case, if x is a neighbor of z, then x must be the 1-th neighbor, in which $z_j z_{j-1} \in \{01, 11\}$ for $j \in \{3, 5, \ldots, n-5\}$ (i.e., $j = 2\ell + 1$ for each $\ell \in [\lceil \frac{n}{2} \rceil - 3]$) and $z_1 z_0 \in \{00, 10\}$. Otherwise, x and z are nonadjacent in FCQ_n.

Similarly, we can derive the representation of y for i odd through the following:

$$
\begin{aligned}
z^i &= 0\ z_{n-2}\ z_{n-3}\ z_{n-4} \cdots z_{i+2}\ z_{i+1}\ \bar{z}_i\ z_{i-1}\ (z_{i-2}\ z_{i-3}) \cdots (z_1\ z_0); \\
\mathrm{ex}(z^i) &= 1\ z_{n-2}\ (z_{n-3}\ z_{n-4}) \cdots (z_{i+2}\ z_{i+1})\ (\bar{z}_i\ z_{i-1})\ z_{i-2}\ z_{i-3} \cdots z_1\ z_0; \\
\mathrm{co}(\mathrm{ex}(z^i)) &= 0\ \bar{z}_{n-2}\ \overline{(z_{n-3}\ z_{n-4})} \cdots \overline{(z_{i+2}\ z_{i+1})}\ \overline{(\bar{z}_i\ z_{i-1})}\ \bar{z}_{i-2}\ \bar{z}_{i-3} \cdots \bar{z}_1\ \bar{z}_0; \\
\mathrm{ex}(\mathrm{co}(\mathrm{ex}(z^i))) &= 1\ \bar{z}_{n-2}\ (\overline{(z_{n-3}\ z_{n-4})}) \cdots (\overline{(z_{i+2}\ z_{i+1})})\ (\overline{(\bar{z}_i\ z_{i-1})})\ (\bar{z}_{i-2}\ \bar{z}_{i-3}) \cdots (\bar{z}_1\ \bar{z}_0); \\
y = \mathrm{co}(\mathrm{ex}(\mathrm{co}(\mathrm{ex}(z^i)))) &= 0\ z_{n-2}\ \overline{(\overline{(z_{n-3}\ z_{n-4})})} \cdots \overline{(\overline{(z_{i+2}\ z_{i+1})})}\ \overline{(\overline{(\bar{z}_i\ z_{i-1})})}\ \overline{(\bar{z}_{i-2}\ \bar{z}_{i-3})} \cdots \overline{(\bar{z}_1\ \bar{z}_0)}.
\end{aligned}
$$

For $j \in \{1, 3, \ldots, i-2\}$, we check $x_j x_{j-1}$ corresponding to $z_j z_{j-1} \in \{00, 01, 11, 10\}$ as follows:

$$x_j x_{j-1} = \overline{(\bar{z}_j \bar{z}_{j-1})} = \begin{cases} \overline{(\bar{0}\bar{0})} = \overline{(11)} = \overline{01} = 10 & \text{if } z_j z_{j-1} = 00; \\ \overline{(\bar{0}\bar{1})} = \overline{(10)} = \overline{10} = 01 & \text{if } z_j z_{j-1} = 01; \\ \overline{(\bar{1}\bar{1})} = \overline{(00)} = \overline{00} = 11 & \text{if } z_j z_{j-1} = 11; \\ \overline{(\bar{1}\bar{0})} = \overline{(01)} = \overline{11} = 00 & \text{if } z_j z_{j-1} = 10. \end{cases} \tag{A.3}$$

As shown above, $x_j x_{j-1} = z_j z_{j-1}$ only when $z_j z_{j-1} \in \{01, 11\}$ and $x_j x_{j-1} \nsim z_j z_{j-1}$ for all $j \in \{1, 3, \ldots, i-2\}$. In addition, we check whether $x_i x_{i-1}$ and $z_i z_{i-1}$ are pair-related as follows:

$$x_i x_{i-1} = \overline{(\overline{(\bar{z}_i z_{i-1})})} = \begin{cases} \overline{(\overline{(\bar{0}0)})} = \overline{(\overline{(10)})} = \overline{(10)} = \overline{(01)} = \overline{11} = 00 & \text{if } z_i z_{i-1} = 00; \\ \overline{(\overline{(\bar{0}1)})} = \overline{(\overline{(11)})} = \overline{(01)} = \overline{(10)} = \overline{10} = 01 & \text{if } z_i z_{i-1} = 01; \\ \overline{(\overline{(\bar{1}1)})} = \overline{(\overline{(01)})} = \overline{(11)} = \overline{(00)} = \overline{00} = 11 & \text{if } z_i z_{i-1} = 11; \\ \overline{(\overline{(\bar{1}0)})} = \overline{(\overline{(00)})} = \overline{(00)} = \overline{(11)} = \overline{01} = 10 & \text{if } z_i z_{i-1} = 10. \end{cases}$$

Clearly, $x_i x_{i-1} = z_i z_{i-1}$ for all cases, and $x_i x_{i-1} \sim z_i z_{i-1}$ appears only when $z_i z_{i-1} \in \{00, 10\}$. Moreover, we check $x_j x_{j-1}$ corresponding to $z_j z_{j-1} \in \{00, 01, 11, 10\}$ for $j \in \{i+2, i+4, \ldots, n-3\}$ as follows:

$$x_j x_{j-1} = \overline{(\overline{(z_j z_{j-1})})} = \begin{cases} \overline{(\overline{(00)})} = \overline{(\overline{00})} = \overline{(11)} = \overline{01} = 10 & \text{if } z_j z_{j-1} = 00; \\ \overline{(\overline{(01)})} = \overline{(\overline{11})} = \overline{(00)} = \overline{00} = 11 & \text{if } z_j z_{j-1} = 01; \\ \overline{(\overline{(11)})} = \overline{(\overline{01})} = \overline{(10)} = \overline{10} = 01 & \text{if } z_j z_{j-1} = 11; \\ \overline{(\overline{(10)})} = \overline{(\overline{10})} = \overline{(01)} = \overline{11} = 00 & \text{if } z_j z_{j-1} = 10. \end{cases} \tag{A.4}$$

This shows that $x_j x_{j-1} \sim z_j z_{j-1}$ appears only when $z_j z_{j-1} \in \{01, 11\}$. A thorough inspection responds that all results for $z_j z_{j-1} \in \{00, 01, 11, 10\}$ are the same for x and y. Thus, a similar argument shows that y and z are adjacent if and only y is the $(n-3)$-th neighbor of z (when $i = 1$) or 1-th neighbor of z (when $i = n-3$) that satisfies $z_j z_{j-1} \in \{01, 11\}$ for $j \in \{3, 5, \ldots, n-5\}$ and $z_1 z_0 \in \{00, 10\}$.

Case 1.2: If i is even, we can derive the representation of x through the following:

$$\begin{aligned} z^i &= 0\ z_{n-2}\ z_{n-3}\ z_{n-4} \cdots z_{i+3}\ z_{i+2}\ z_{i+1}\ \bar{z}_i\ (z_{i-1}\ z_{i-2}) \cdots (z_1\ z_0); \\ \mathrm{co}(z^i) &= 1\ \bar{z}_{n-2}\ \bar{z}_{n-3}\ \bar{z}_{n-4} \cdots \bar{z}_{i+3}\ \bar{z}_{i+2}\ \bar{z}_{i+1}\ z_i\ \overline{(z_{i-1}\ z_{i-2})} \cdots \overline{(z_1\ z_0)}; \\ \mathrm{ex}(\mathrm{co}(z^i)) &= 0\ \bar{z}_{n-2}\ (\bar{z}_{n-3}\bar{z}_{n-4}) \cdots (\bar{z}_{i+3}\bar{z}_{i+2})\ (\bar{z}_{i+1}z_i)\ (\overline{(z_{i-1}\ z_{i-2})}) \cdots (\overline{(z_1\ z_0)}); \\ \mathrm{co}(\mathrm{ex}(\mathrm{co}(z^i))) &= 1\ z_{n-2}\ \overline{(\bar{z}_{n-3}\bar{z}_{n-4})} \cdots \overline{(\bar{z}_{i+3}\bar{z}_{i+2})}\ \overline{(\bar{z}_{i+1}z_i)}\ \overline{(\overline{(z_{i-1}\ z_{i-2})})} \cdots \overline{(\overline{(z_1\ z_0)})}; \\ x = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(\mathrm{co}(z^i)))) &= 0\ z_{n-2}(\overline{(\bar{z}_{n-3}\bar{z}_{n-4})}) \cdots (\overline{(\bar{z}_{i+3}\bar{z}_{i+3})})(\overline{(\bar{z}_{i+1}z_i)})(\overline{(\overline{(z_{i-1}\ z_{i-2})})}) \cdots (\overline{(\overline{(z_1\ z_0)})}). \end{aligned}$$

We observe that the representation of $(\overline{(\overline{(z_j z_{j-1})})})$ for $j \in \{1, 3, \ldots, i-1\}$ is the same as Eq. (A.1), and the representation of $(\overline{(\bar{z}_j \bar{z}_{j-1})})$ for $j \in \{i+3, i+5, \ldots, n-3\}$ is the same as Eq. (A.2). Also, we check $x_{i+1} x_i$ corresponding to

$z_{i+1}z_i \in \{00, 01, 11, 10\}$ as follows:

$$x_i x_{i-1} = (\,\overline{(\bar{z}_{i+1} z_i)}\,) = \begin{cases} (\,\overline{(\bar{0}0)}\,) = (\,\overline{(10)}\,) = (\,\overline{10}\,) = (01) = 11 & \text{if } z_i z_{i-1} = 00; \\ (\,\overline{(\bar{0}1)}\,) = (\,\overline{(11)}\,) = (\,\overline{01}\,) = (10) = 10 & \text{if } z_i z_{i-1} = 01; \\ (\,\overline{(\bar{1}1)}\,) = (\,\overline{(01)}\,) = (\,\overline{11}\,) = (00) = 00 & \text{if } z_i z_{i-1} = 11; \\ (\,\overline{(\bar{1}0)}\,) = (\,\overline{(00)}\,) = (\,\overline{00}\,) = (11) = 01 & \text{if } z_i z_{i-1} = 10. \end{cases} \tag{A.5}$$

As $x_{i+1}x_i = \overline{z_{i+1}z_i}$ and $x_{i+1}x_i \nsim z_{i+1}z_i$ for all cases, x is a not neighbor of z in this case.

Similarly, we can derive the representation of y for i even through the following:

$$\begin{aligned} z^i &= 0\; z_{n-2}\; z_{n-3}\, z_{n-4} \cdots z_{i+3}\; z_{i+2}\; z_{i+1}\; \bar{z}_i\; (z_{i-1}\; z_{i-2}) \cdots (z_1\; z_0); \\ \mathrm{ex}(z^i) &= 1\; z_{n-2}\; (z_{n-3}\, z_{n-4}) \cdots (z_{i+3}\; z_{i+2})\, (z_{i+1}\; \bar{z}_i)\; z_{i-1}\; z_{i-2} \cdots z_1\; z_0; \\ \mathrm{co}(\mathrm{ex}(z^i)) &= 0\; \bar{z}_{n-2}\; \overline{(z_{n-3}\, z_{n-4})} \cdots \overline{(z_{i+3}\; z_{i+2})}\; \overline{(z_{i+1}\; \bar{z}_i)}\; \bar{z}_{i-1}\; \bar{z}_{i-2} \cdots \bar{z}_1\; \bar{z}_0; \\ \mathrm{ex}(\mathrm{co}(\mathrm{ex}(z^i))) &= 1\; \bar{z}_{n-2}\; (\,\overline{(z_{n-3}\, z_{n-4})}\,) \cdots (\,\overline{(z_{i+3}\; z_{i+2})}\,)\; (\,\overline{(z_{i+1}\; \bar{z}_i)}\,)\; (\bar{z}_{i-1}\; \bar{z}_{i-2}) \cdots (\bar{z}_1\; \bar{z}_0); \\ y = \mathrm{co}(\mathrm{ex}(\mathrm{co}(\mathrm{ex}(z^i)))) &= 0\; z_{n-2}\; \overline{(\,\overline{(z_{n-3}\, z_{n-4})}\,)} \cdots \overline{(\,\overline{(z_{i+3}\; z_{i+2})}\,)}\; \overline{(\,\overline{(z_{i+1}\; \bar{z}_i)}\,)}\; \overline{(\bar{z}_{i-1}\; \bar{z}_{i-2})} \cdots \overline{(\bar{z}_1\; \bar{z}_0)}. \end{aligned}$$

We observe that the representation of $\overline{(\bar{z}_j \bar{z}_{j-1})}$ for $j \in \{1, 3, \ldots, i-1\}$ is the same as Eq. (A.3), and the representation of $(\,\overline{(z_j z_{j-1})}\,)$ for $j \in \{i+3, i+5, \ldots, n-3\}$ is the same as Eq. (A.4). Also, we check $x_{i+1}x_i$ corresponding to $z_{i+1}z_i \in \{00, 01, 11, 10\}$ as follows:

$$\begin{aligned} x_i x_{i-1} &= \overline{(\,\overline{(z_{i+1}\; \bar{z}_i)}\,)} \\ &= \begin{cases} \overline{(\overline{(0\bar{0})})} = \overline{(\overline{(01)})} = \overline{(\overline{11})} = \overline{(00)} = \overline{00} = 11 & \text{if } z_i z_{i-1} = 00; \\ \overline{(\overline{(0\bar{1})})} = \overline{(\overline{(00)})} = \overline{(\overline{00})} = \overline{(11)} = \overline{01} = 10 & \text{if } z_i z_{i-1} = 01; \\ \overline{(\overline{(1\bar{1})})} = \overline{(\overline{(10)})} = \overline{(\overline{10})} = \overline{(01)} = \overline{11} = 00 & \text{if } z_i z_{i-1} = 11; \\ \overline{(\overline{(1\bar{0})})} = \overline{(\overline{(11)})} = \overline{(\overline{01})} = \overline{(10)} = \overline{10} = 01 & \text{if } z_i z_{i-1} = 10. \end{cases} \end{aligned} \tag{A.6}$$

As $x_{i+1}x_i = \overline{z_{i+1}z_i}$ and $x_{i+1}x_i \nsim z_{i+1}z_i$ for all cases, x is a not neighbor of z in this case.

Case 2: n is odd. The case of odd n can be proved similarly, which we omit here. □

Appendix B: Proof of Lemma 3.3

Let $z = 0z_{n-2} \cdots z_1 z_0$ be any vertex in L_0. Clearly, $\mathrm{ex}(z)$ and $\mathrm{co}(z)$ are two neighbors of z located in L_1. Precisely, $N_{L_1}(z) = \{\mathrm{ex}(z), \mathrm{co}(z)\}$. As $\mathrm{ex}(z) \in V(F)$ and $c(F) \in V(L_0)\backslash\{z\}$, it implies $c(F) = \mathrm{co}(\mathrm{ex}(z))$. Similarly, as $\mathrm{co}(z) \in V(F')$ and $c(F') \in V(L_0)\backslash\{z\}$, it follows $c(F') = \mathrm{ex}(\mathrm{co}(z))$. We now verify that $\mathrm{co}(\mathrm{ex}(z)) \neq \mathrm{ex}(\mathrm{co}(z))$

as follows:

$$\mathrm{co}(\mathrm{ex}(z)) = \begin{cases} 0\,\bar{z}_{n-2}\,\overline{(z_{n-3}\,z_{n-4})}\cdots\overline{(z_j\,z_{j-1})}\cdots\overline{(z_1\,z_0)} & \text{if } n \text{ is even;} \\ 0\,\overline{(z_{n-2}\,z_{n-3})}\cdots\overline{(z_j\,z_{j-1})}\cdots\overline{(z_1\,z_0)} & \text{if } n \text{ is odd} \end{cases}$$

and

$$\mathrm{ex}(\mathrm{co}(z)) = \begin{cases} 0\,\bar{z}_{n-2}\,(\bar{z}_{n-3}\,\bar{z}_{n-4})\cdots(\bar{z}_j\,\bar{z}_{j-1})\cdots(\bar{z}_1\,\bar{z}_0) & \text{if } n \text{ is even;} \\ 0\,(\bar{z}_{n-2}\,\bar{z}_{n-3})\cdots(\bar{z}_j\,\bar{z}_{j-1})\cdots(\bar{z}_1\,\bar{z}_0) & \text{if } n \text{ is odd.} \end{cases}$$

where $j = 2h + 1$ for $h \in [\lceil \frac{n}{2} \rceil - 2]$. For each $z_j z_{j-1} \in \{00, 01, 11, 10\}$, we have

$$\overline{(z_j\,z_{j-1})} = \begin{cases} \overline{(00)} = \overline{00} = 11 & \text{if } z_j z_{j-1} = 00; \\ \overline{(01)} = \overline{11} = 00 & \text{if } z_j z_{j-1} = 01; \\ \overline{(11)} = \overline{01} = 10 & \text{if } z_j z_{j-1} = 11; \\ \overline{(10)} = \overline{10} = 01 & \text{if } z_j z_{j-1} = 10 \end{cases}$$

and

$$(\bar{z}_j\,\bar{z}_{j-1}) = \begin{cases} (\bar{0}\,\bar{0}) = (11) = 01 & \text{if } z_j z_{j-1} = 00; \\ (\bar{0}\,\bar{1}) = (10) = 10 & \text{if } z_j z_{j-1} = 01; \\ (\bar{1}\,\bar{1}) = (00) = 00 & \text{if } z_j z_{j-1} = 11; \\ (\bar{1}\,\bar{0}) = (01) = 11 & \text{if } z_j z_{j-1} = 10. \end{cases}$$

It is easy to check that $\overline{(z_j\ z_{j-1})} \neq (\bar{z}_j\ \bar{z}_{j-1})$ for each $z_j z_{j-1} \in \{00, 01, 11, 10\}$. Thus, $c(F) = \mathrm{co}(\mathrm{ex}(z)) \neq \mathrm{ex}(\mathrm{co}(z)) = c(F')$. □

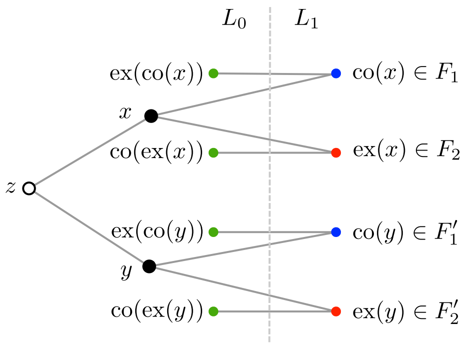
Appendix C: Proof of Lemma 3.4

By Lemma 3.3, $c(F_1) \neq c(F_2)$ and $c(F_1') \neq c(F_2')$. Since $\mathrm{co}(x) \in V(F_1)$ and $\mathrm{ex}(x) \in V(F_2)$ are located in L_1 and $c(F_1), c(F_2) \in V(L_0)\backslash\{z\}$, either $c(F_1) \in \{x, \mathrm{ex}(\mathrm{co}(x))\}$ and $c(F_2) = \mathrm{co}(\mathrm{ex}(x))$ or $c(F_2) \in \{x, \mathrm{co}(\mathrm{ex}(x))\}$ and $c(F_1) = \mathrm{ex}(\mathrm{co}(x))$. See Fig. 7 for illustration.

(1) By symmetry, we only need to prove that $c(F_1) \neq c(F_1')$, $c(F_2) \neq c(F_2')$, and $c(F_1) \neq c(F_2')$. Suppose on the contrary, we consider the following cases:

Case 1: $c(F_1) = c(F_1')$. As $\mathrm{co}(y) \in V(F_1')$, if $c(F_1') = c(F_1) = x$, then either $\mathrm{co}(y) = \mathrm{co}(x)$ or $\mathrm{co}(y) = \mathrm{ex}(x)$. The former implies $x = y$, which contradicts the fact that x and y are distinct. For the latter case, by Lemma 2.4, since $n \geq 5$ and $\bar{y} = \mathrm{co}(y) = \mathrm{ex}(x) = x^{n-1}$, we have $\mathrm{dist}_{L_0}(x, y) \geq \lceil \frac{n}{2} \rceil = 3$, which contradicts the fact that z is a common neighbor of x and y in L_0. On the other hand, as $\mathrm{co}(y) \in V(F_1')$, if $c(F_1') = c(F_1) = \mathrm{ex}(\mathrm{co}(x))$, then either $\mathrm{co}(y) = \mathrm{co}(x)$ or $\mathrm{co}(y) = \mathrm{co}(\mathrm{ex}(\mathrm{co}(x)))$. The former again implies $x = y$, a contradiction. The latter implies $y = \mathrm{ex}(\mathrm{co}(x))$ and $\mathrm{ex}(y) = \mathrm{co}(x)$. By Lemma 2.4, since $n \geq 5$ and $\bar{x} = \mathrm{co}(x) = \mathrm{ex}(y) = y^{n-1}$, we have $\mathrm{dist}_{L_0}(x, y) \geq \lceil \frac{n}{2} \rceil = 3$, which contradicts the fact that $x, y \in N_{L_0}(z)$.

Fig. 7 Illustration of Lemma 3.4



Case 2: $c(F_2) = c(F_2')$. As $\mathrm{ex}(y) \in V(F_2')$, if $c(F_2') = c(F_2) = x$, then either $\mathrm{ex}(y) = \mathrm{ex}(x)$ or $\mathrm{ex}(y) = \mathrm{co}(x)$. The former implies $x = y$, which contradicts the fact that x and y are distinct. For the latter case, by Lemma 2.4, since $n \geq 5$ and $\bar{x} = \mathrm{co}(x) = \mathrm{ex}(y) = y^{n-1}$, we have $\mathrm{dist}_{L_0}(x, y) \geq \lceil \frac{n}{2} \rceil = 3$, which contradicts the fact that z is a common neighbor of x and y in L_0. On the other hand, as $\mathrm{ex}(y) \in V(F_2')$, if $c(F_2') = c(F_2) = \mathrm{co}(\mathrm{ex}(x))$, then either $\mathrm{ex}(y) = \mathrm{ex}(x)$ or $\mathrm{ex}(y) = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(x)))$. The former again implies $y = x$, a contradiction. The latter implies $y = \mathrm{co}(\mathrm{ex}(x))$ and $\mathrm{co}(y) = \mathrm{ex}(x)$. By Lemma 2.4, since $n \geq 5$ and $\bar{y} = \mathrm{co}(y) = \mathrm{ex}(x) = x^{n-1}$, we have $\mathrm{dist}_{L_0}(x, y) \geq \lceil \frac{n}{2} \rceil = 3$, which contradicts the fact that $x, y \in N_{L_0}(z)$.

Case 3: $c(F_1) = c(F_2')$. As $\mathrm{ex}(y) \in V(F_2')$ located in L_1, if $c(F_2') = c(F_1) = x$, then either $\mathrm{ex}(y) = \mathrm{ex}(x)$ or $\mathrm{ex}(y) = \mathrm{co}(x)$. Then, a proof can derive the same contradiction as Case 2. On the other hand, as $\mathrm{ex}(y) \in V(F_2')$ located in L_1, if $c(F_2') = c(F_1) = \mathrm{ex}(\mathrm{co}(x))$ and $\mathrm{ex}(y) \neq \mathrm{co}(x)$, then $\mathrm{ex}(y) = \mathrm{co}(\mathrm{ex}(\mathrm{co}(x)))$. Thus, $y = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(\mathrm{co}(x))))$. By Lemma 3.2 and one of the conditions a, b, and c in the assertion (1), $y = \mathrm{ex}(\mathrm{co}(\mathrm{ex}(\mathrm{co}(x))))$ is not a neighbor of z, a contradiction.

(2) For n even, without loss of generality, assume $x = z^1$ and $y = z^{n-3}$. Since $\mathrm{co}(x) \in V(F_1)$, $\mathrm{ex}(y) \in V(F_2')$, and $c(F_1), c(F_2') \notin \{x, y\}$, one has that

$$c(F_1) = \mathrm{ex}(\mathrm{co}(x)) = \mathrm{ex}(\mathrm{co}(z^1)) = 0\bar{z}_{n-2}\,(\bar{z}_{n-3}\,\bar{z}_{n-4})\,(\bar{z}_{n-5}\,\bar{z}_{n-6}) \cdots (\bar{z}_3\,\bar{z}_2)\,(z_1\,\bar{z}_0).$$

and

$$c(F_2') = \mathrm{co}(\mathrm{ex}(y)) = \mathrm{co}(\mathrm{ex}(z^{n-3})) = 0\bar{z}_{n-2}\,\overline{(\bar{z}_{n-3}\,z_{n-4})}\,\bar{z}_{n-5}\,\bar{z}_{n-6} \cdots \bar{z}_1\,\bar{z}_0.$$

We now check the representations of $c(F_1)$ and $c(F_2')$ as follows. Clearly,

$$(\bar{z}_{n-3}\,\bar{z}_{n-4}) = \begin{cases} (\bar{0}\bar{0}) = (11) = 01 = \overline{10} = \overline{(10)} = \overline{(\bar{0}\bar{0})} = \overline{(\bar{z}_{n-3} z_{n-4})} & \text{if } z_{n-3}z_{n-4} = 00; \\ (\bar{0}\bar{1}) = (10) = 10 = \overline{01} = \overline{(11)} = \overline{(\bar{0}1)} = \overline{(\bar{z}_{n-3} z_{n-4})} & \text{if } z_{n-3}z_{n-4} = 01; \\ (\bar{1}\bar{1}) = (00) = 00 = \overline{11} = \overline{(01)} = \overline{(\bar{1}1)} = \overline{(\bar{z}_{n-3} z_{n-4})} & \text{if } z_{n-3}z_{n-4} = 11; \\ (\bar{1}\bar{0}) = (01) = 11 = \overline{00} = \overline{(00)} = \overline{(\bar{1}0)} = \overline{(\bar{z}_{n-3} z_{n-4})} & \text{if } z_{n-3}z_{n-4} = 10. \end{cases} \tag{A.7}$$

Thus, $(\bar{z}_{n-3}\bar{z}_{n-4}) = \overline{(\bar{z}_{n-3}z_{n-4})}$. For each $\ell \in [\lceil \frac{n}{2} \rceil - 3]$ with $j = 2\ell + 1$, we have

$$(\bar{z}_j\, \bar{z}_{j-1}) = \begin{cases} (\bar{0}\bar{1}) = (10) = 10 = \bar{0}\bar{1} = \bar{z}_j\bar{z}_{j-1} & \text{if } z_j z_{j-1} = 01; \\ (\bar{1}\bar{1}) = (00) = 00 = \bar{1}\bar{1} = \bar{z}_j\bar{z}_{j-1} & \text{if } z_j z_{j-1} = 11. \end{cases}$$

Thus, $(\bar{z}_j\bar{z}_{j-1}) = \bar{z}_j\bar{z}_{j-1}$ for $z_j z_{j-1} \in \{01, 11\}$. Also, we have

$$(z_1\, \bar{z}_0) = \begin{cases} (0\bar{0}) = (01) = 11 = \bar{0}\bar{0} = \bar{z}_1\bar{z}_0 & \text{if } z_1 z_0 = 00; \\ (1\bar{0}) = (11) = 01 = \bar{1}\bar{0} = \bar{z}_1\bar{z}_0 & \text{if } z_1 z_0 = 10. \end{cases}$$

Thus, $(z_1\bar{z}_0) = \bar{z}_1\bar{z}_0$ for $z_1 z_0 \in \{00, 10\}$. Therefore, if $z_j z_{j-1} \in \{01, 11\}$ where $j = 2\ell + 1$ for each $\ell \in [\lceil \frac{n}{2} \rceil - 3]$ and $z_1 z_0 \in \{00, 10\}$, then $c(F_1) = c(F_2')$.

Furthermore, since $\mathrm{ex}(x) \in V(F_2)$, $\mathrm{co}(y) \in V(F_1')$, and $c(F_2), c(F_1') \notin \{x, y\}$, one has that

$$c(F_2) = \mathrm{co}(\mathrm{ex}(x)) = \mathrm{co}(\mathrm{ex}(z^1)) = 0\bar{z}_{n-2}\, \overline{(z_{n-3}\, z_{n-4})}\, \overline{(z_{n-5}\, z_{n-6})} \cdots \overline{(z_3\, z_2)}\, \overline{(\bar{z}_1\, z_0)}$$

and

$$\begin{aligned} c(F_1') = \mathrm{ex}(\mathrm{co}(y)) &= \mathrm{ex}(\mathrm{co}(z^{n-3})) = 0\bar{z}_{n-2}\, (z_{n-3}\, \bar{z}_{n-4})\, (\overline{(z_{n-5}\, z_{n-6})}) \\ &\cdots (\overline{(z_3\, z_2)})\, (\overline{(z_1\, z_0)}). \end{aligned}$$

We now check the representations of $c(F_2)$ and $c(F_1')$ as follows. Taking the complement of z_{n-3} in Eq. (A.7), we can obtain $(z_{n-3}\bar{z}_{n-4}) = \overline{(z_{n-3}z_{n-4})}$. For each $\ell \in [\lceil \frac{n}{2} \rceil - 3]$ with $j = 2\ell + 1$, we have

$$\overline{(z_j\, z_{j-1})} = \begin{cases} \overline{(01)} = \overline{11} = 00 = (00) = (\overline{(z_j\, z_{j-1})}) & \text{if } z_j z_{j-1} = 01; \\ \overline{(11)} = \overline{01} = 10 = (10) = (\overline{(z_j\, z_{j-1})}) & \text{if } z_j z_{j-1} = 11. \end{cases}$$

Thus, $\overline{(z_j\, z_{j-1})} = (\overline{(z_j\, z_{j-1})})$ for $z_j z_{j-1} \in \{01, 11\}$. Also, we have

$$\overline{(\bar{z}_1\, z_0)} = \begin{cases} \overline{(\bar{0}0)} = \overline{(10)} = \overline{10} = 01 = (11) = (\overline{00}) = (\overline{(00)}) = (\overline{(z_1\, z_0)}) & \text{if } z_1 z_0 = 00; \\ \overline{(\bar{1}0)} = \overline{(00)} = \overline{00} = 11 = (01) = (\overline{10}) = (\overline{(10)}) = (\overline{(z_1\, z_0)}) & \text{if } z_1 z_0 = 10. \end{cases}$$

Thus, $\overline{(\bar{z}_1 z_0)} = (\overline{(z_1 z_0)})$ for $z_1 z_0 \in \{00, 10\}$. Therefore, if $z_j z_{j-1} \in \{01, 11\}$ where $j = 2\ell + 1$ for each $\ell \in [\lceil \frac{n}{2} \rceil - 3]$ and $z_1 z_0 \in \{00, 10\}$, then $c(F_2) = c(F_1')$.

The case of odd n can be proved similarly, which we omit here.

□

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